1.1. The Gaussian Integral

Basic Form

Let

$$\mathcal{Z}(\boldsymbol{A}) = \int_{-\infty}^{\infty} d\,x_1 \;\ldots \int_{-\infty}^{\infty} d\,x_n\, e^{-A_2(\boldsymbol{x})} \equiv \int d^n\,x\, e^{-A_2(\boldsymbol{x})} \tag{1.1}$$

where $\boldsymbol{A}$ is a real, symmetric matrix &

$$A_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x} = \frac{1}{2}\sum_{i,j=1}^{n} x_i\, A_{i\,j}\, x_j \tag{1.2}$$

$\boldsymbol{A}$ is real & symmetric

→ $\boldsymbol{A}$ is diagonalizable by a orthogonal transformation, i.e., there exists

$$\boldsymbol{O}^{-1} = \boldsymbol{O}^T$$

such that

$$\boldsymbol{O}\,\boldsymbol{A}\,\boldsymbol{O}^T = \boldsymbol{\lambda} = \mathrm{diag}(\lambda_i)$$

Note that λ_i are all real.

$$\therefore \qquad A_2(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{O}^T\boldsymbol{\lambda}\,\boldsymbol{O}\,\boldsymbol{x} = \frac{1}{2}y^T\boldsymbol{\lambda}\,\boldsymbol{y} = \frac{1}{2}\sum_{i=1}^{n}\lambda_i\, y_i^2$$

where

$$\boldsymbol{y} = \boldsymbol{O}\,\boldsymbol{x}$$

$$d^n\,x = \frac{\partial\,(y_1,\ \ldots,\ y_n)}{\partial\,(x_1,\ \ldots,\ x_n)}\,d^n\,y = \det|\,\boldsymbol{O}\,|\;d^n\,y = d^n\,y$$

$$\rightarrow \qquad \mathcal{Z}(\boldsymbol{A}) = \int d^n\,y\,\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\lambda_i\, y_i^2\right)$$

$$= \prod_{i=1}^{n}\int_{-\infty}^{\infty} d\,y_i\,\exp\left(-\frac{1}{2}\lambda_i\, y_i^2\right)$$

Provided $\boldsymbol{A}$ is positive definite, i.e.,

$$\lambda_i > 0 \qquad \forall\, i$$

the integrals are well defined.

Using

$$\int_{-\infty}^{\infty} d\,y\,\exp\left(-\frac{1}{2}a\,y^2\right) = \sqrt{\frac{2\,\pi}{a}}$$

we have

$$\mathcal{Z}(\boldsymbol{A}) = \prod_{i=1}^{n}\sqrt{\frac{2\,\pi}{\lambda_i}} = (2\,\pi)^{n/2}\,\frac{1}{\sqrt{\prod_{i=1}^{n}\lambda_i}} = \frac{(2\,\pi)^{n/2}}{\sqrt{\det\boldsymbol{A}}} \tag{1.3}$$

To summarize, for a real, symmetric & positive definite $\boldsymbol{A}$,

$$\int d^n\,x\,\exp\left(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\,\boldsymbol{x}\right) = \frac{(2\,\pi)^{n/2}}{\sqrt{\det\boldsymbol{A}}}$$

If $\boldsymbol{A}$ is hermitian & positive definite, then it can be diagonalized by a unitary transformation

$$\boldsymbol{U}\,\boldsymbol{A}\,\boldsymbol{U}^{+} = \boldsymbol{\lambda} = \mathrm{diag}(\lambda_i) \qquad \text{with} \qquad \lambda_i > 0 \qquad \forall\, i$$

$$\to \qquad \int d^n x \exp\left(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x}\right) = J \frac{(2\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}}$$

where $|\,J\,| = |\det \boldsymbol{U}\,| = 1$.

If $\boldsymbol{A}$ is normal & positive definite, then it can be diagonalized by a similarity transformation

$$\boldsymbol{S}\,\boldsymbol{A}\,\boldsymbol{S}^{-1} = \boldsymbol{\lambda} = \mathrm{diag}(\lambda_i) \qquad \text{with} \qquad \lambda_i > 0 \qquad \forall\, i$$

$$\to \qquad \int d^n x \exp\left(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x}\right) = J \frac{(2\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}}$$

where $J = \det \boldsymbol{S}$.

General Form

General form of a Gaussian integral is:

$$\mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) = \int d^n x\, e^{-J(\boldsymbol{A},\boldsymbol{b}\,;\boldsymbol{x})} \qquad (1.4)$$

where

$$\begin{aligned} J(\boldsymbol{A}, \boldsymbol{b}\,; \boldsymbol{x}) &= \boldsymbol{A}_2(\boldsymbol{x}) - \boldsymbol{b}^T \boldsymbol{x} \\ &= \frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x} \\ &= \frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j - \sum_{i=1}^{n} b_i x_i \end{aligned}$$

Minima of J satisfy

$$\begin{aligned} \frac{\partial J}{\partial x_k} = 0 \qquad\qquad & \forall\, k \\ = \frac{1}{2}\left(\sum_{j=1}^{n} A_{kj} x_j + \sum_{i=1}^{n} x_i A_{ik}\right) - b_k & \\ = \sum_{j=1}^{n} A_{kj} x_j - b_k & \end{aligned}$$

$$\lambda_i > 0 \quad \to \qquad \det \boldsymbol{A} \neq 0$$

$$\to \qquad \boldsymbol{A}^{-1} \text{ exists}$$

$\therefore$ Minima of J are given by

$$\boldsymbol{A}\,\boldsymbol{x}^0 = \boldsymbol{b}$$

$$\to \qquad \boldsymbol{x}^0 = \boldsymbol{A}^{-1}\,\boldsymbol{b}$$

$$x_i^0 = \sum_{j=1}^{n} (\boldsymbol{A}^{-1})_{ij}\, b_j \qquad (1.5)$$

Let

$$\boldsymbol{x} = \boldsymbol{x}^0 + \boldsymbol{y} = \boldsymbol{A}^{-1}\,\boldsymbol{b} + \boldsymbol{y} \qquad (1.6)$$

$\to$

$$d^n x = d^n y$$

Assuming $\boldsymbol{A}^T = \boldsymbol{A}$,

$$J = \frac{1}{2}\left(\boldsymbol{b}^T \boldsymbol{A}^{-1} + \boldsymbol{y}^T\right) \boldsymbol{A} \left(\boldsymbol{A}^{-1}\, \boldsymbol{b} + \boldsymbol{y}\right) - \boldsymbol{b}^T\left(\boldsymbol{A}^{-1}\, \boldsymbol{b} + \boldsymbol{y}\right)$$
$$= \frac{1}{2}\left(\boldsymbol{b}^T \boldsymbol{A}^{-1}\, \boldsymbol{b} + \boldsymbol{b}^T \boldsymbol{y} + \boldsymbol{y}^T \boldsymbol{b} + \boldsymbol{y}^T \boldsymbol{A}\, \boldsymbol{y}\right) - \boldsymbol{b}^T \boldsymbol{A}^{-1}\, \boldsymbol{b} - \boldsymbol{b}^T \boldsymbol{y}$$

$$\boldsymbol{b}^T \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{b} = \sum_{i=1}^{n} b_i\, y_i$$

$$\rightarrow \qquad J = -\frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1}\, \boldsymbol{b} + \frac{1}{2} \boldsymbol{y}^T \boldsymbol{A}\, \boldsymbol{y}$$
$$= -w_2(\boldsymbol{b}) + A_2(\boldsymbol{y})$$

where

$$w_2(\boldsymbol{b}) = \frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1}\, \boldsymbol{b} \tag{1.7}$$

Hence,

$$\mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) = e^{w_2(\boldsymbol{b})} \int d^n\, y\, e^{-A_2(\boldsymbol{y})}$$
$$= e^{w_2(\boldsymbol{b})}\, \mathcal{Z}(\boldsymbol{A})$$
$$= \frac{(2\,\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}} \exp\left(\frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1}\, \boldsymbol{b}\right) \tag{1.8}$$

Note that eq(1.8) is applicable only if $\boldsymbol{A}$ is real, symmetric & positive definite.

Gaussian Expectation Values

$$\langle x_{k_1}\, x_{k_2} \ldots x_{k_l} \rangle \equiv \frac{1}{\mathcal{Z}(\boldsymbol{A})} \int d^n\, x\, e^{-A_2(\boldsymbol{x})}\, x_{k_1}\, x_{k_2} \ldots x_{k_l} \tag{1.9}$$

so that

$$\langle 1 \rangle \equiv \frac{1}{\mathcal{Z}(\boldsymbol{A})} \int d^n\, x\, e^{-A_2(\boldsymbol{x})} = 1$$

$$\mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) = \int d^n\, x\, e^{-A_2(\boldsymbol{x}) + \boldsymbol{b}^T \boldsymbol{x}}$$

$$\rightarrow \qquad \frac{\partial}{\partial\, b_k} \mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) = \int d^n\, x\, e^{-A_2(\boldsymbol{x}) + \boldsymbol{b}^T \boldsymbol{x}}\, x_k$$

$$\therefore \qquad \langle x_k \rangle = \frac{1}{\mathcal{Z}(\boldsymbol{A})} \frac{\partial}{\partial\, b_k} \mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) \Big|_{\boldsymbol{b}=0}$$
$$= \frac{\sqrt{\det \boldsymbol{A}}}{(2\,\pi)^{n/2}} \frac{\partial}{\partial\, b_k} \mathcal{Z}(\boldsymbol{A}, \boldsymbol{b}) \Big|_{\boldsymbol{b}=0}$$

Similarly,

$$\langle x_{k_1}\, x_{k_2} \ldots x_{k_l} \rangle = \frac{1}{\mathcal{Z}(\boldsymbol{A})} \left[\frac{\partial}{\partial\, b_{k_1}} \ldots \frac{\partial}{\partial\, b_{k_l}} \mathcal{Z}(\boldsymbol{A}, \boldsymbol{b})\right] \Big|_{\boldsymbol{b}=0}$$
$$= \frac{\sqrt{\det \boldsymbol{A}}}{(2\,\pi)^{n/2}} \left[\frac{\partial}{\partial\, b_{k_1}} \ldots \frac{\partial}{\partial\, b_{k_l}} \mathcal{Z}(\boldsymbol{A}, \boldsymbol{b})\right] \Big|_{\boldsymbol{b}=0} \tag{1.10}$$

$$\mathcal{Z}\,(\boldsymbol{A}, \boldsymbol{b}) = e^{w_2(\boldsymbol{b})}\, \mathcal{Z}(\boldsymbol{A})$$

$$\rightarrow \qquad \langle x_{k_1} x_{k_2} \ldots x_{k_l} \rangle = \left[\frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_l}} e^{w_2(\boldsymbol{b})} \right] \Bigg|_{\boldsymbol{b}=0} \tag{1.11}$$

For any analytic function

$$F(\boldsymbol{x}) = \sum_{k=0}^{\infty} \left(\frac{d^k F}{d x^k} \right)_{x=0} \frac{x^k}{k!}$$

we have

$$\begin{aligned} \langle F(\boldsymbol{x}) \rangle &= \sum_{k=0}^{\infty} \left(\frac{d^k F}{d x^k} \right)_{x=0} \frac{\langle x^k \rangle}{k!} \\ &= \sum_{k=0}^{\infty} \left(\frac{d^k F}{d x^k} \right)_{x=0} \frac{1}{k!} \left(\frac{\partial}{\partial b} \right)^k e^{w_2(\boldsymbol{b})} \Bigg|_{\boldsymbol{b}=0} \\ &= F\left(\frac{\partial}{\partial b} \right) e^{w_2(\boldsymbol{b})} \Bigg|_{\boldsymbol{b}=0} \end{aligned} \tag{1.12}$$

Wick's Theorem

$$w_2(\boldsymbol{b}) = \frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b} = \frac{1}{2} \sum_{i,j=1}^{n} b_i A_{ij}^{-1} b_j$$

$$\begin{aligned} \rightarrow \qquad \frac{\partial}{\partial b_k} w_2(\boldsymbol{b}) &= \frac{1}{2} \left(\sum_{j=1}^{n} A_{kj}^{-1} b_j + \sum_{i=1}^{n} b_i A_{ik}^{-1} \right) \\ &= \sum_{j=1}^{n} A_{kj}^{-1} b_j \qquad (\ \boldsymbol{A} \text{ is symmetric }) \end{aligned}$$

$$\therefore \qquad \frac{\partial}{\partial b_k} e^{w_2(\boldsymbol{b})} = \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} e^{w_2(\boldsymbol{b})}$$

$$\langle x_k \rangle = \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} e^{w_2(\boldsymbol{b})} \Bigg|_{\boldsymbol{b}=0} = 0$$

$$\begin{aligned} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_k} e^{w_2(\boldsymbol{b})} &= \frac{\partial}{\partial b_j} \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} e^{w_2(\boldsymbol{b})} \\ &= \left(A_{kj}^{-1} + \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} \sum_{j'=1}^{n} A_{jj'}^{-1} b_{j'} \right) e^{w_2(\boldsymbol{b})} \end{aligned}$$

$$\rightarrow \qquad \langle x_j x_k \rangle = A_{kj}^{-1} = A_{jk}^{-1}$$

$$\begin{aligned} \frac{\partial}{\partial b_m} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_k} e^{w_2(\boldsymbol{b})} = \Bigg[& \left(A_{kj}^{-1} + \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} \sum_{j'=1}^{n} A_{jj'}^{-1} b_{j'} \right) \sum_{m'=1}^{n} A_{mm'}^{-1} b_{m'} \\ & + A_{km}^{-1} \sum_{j'=1}^{n} A_{jj'}^{-1} b_{j'} + \sum_{k'=1}^{n} A_{kk'}^{-1} b_{k'} A_{jm}^{-1} \Bigg] e^{w_2(\boldsymbol{b})} \end{aligned}$$

$$\rightarrow \qquad \langle x_m x_j x_k \rangle = 0$$

$$\frac{\partial}{\partial b_p}\frac{\partial}{\partial b_m}\frac{\partial}{\partial b_j}\frac{\partial}{\partial b_k}e^{w_2(\boldsymbol{b})} = \Bigg\{ \Bigg[\left(A^{-1}_{k\,j} + \sum_{k'=1}^{n} A^{-1}_{k\,k'}\,b_{k'} \sum_{j'=1}^{n} A^{-1}_{j\,j'}\,b_{j'}\right) \sum_{m'=1}^{n} A^{-1}_{m\,m'}\,b_{m'}$$

$$+ A^{-1}_{k\,m} \sum_{j'=1}^{n} A^{-1}_{j\,j'}\,b_{j'} + \sum_{k'=1}^{n} A^{-1}_{k\,k'}\,b_{k'}\,A^{-1}_{j\,m} \Bigg] \sum_{p'=1}^{n} A^{-1}_{p\,p'}\,b_{p'}$$

$$+ \left(A^{-1}_{k\,p} \sum_{j'=1}^{n} A^{-1}_{j\,j'}\,b_{j'} + \sum_{k'=1}^{n} A^{-1}_{k\,k'}\,b_{k'}\,A^{-1}_{j\,p'}\right) \sum_{m'=1}^{n} A^{-1}_{m\,m'}\,b_{m'}$$

$$+ \left(A^{-1}_{k\,j} + \sum_{k'=1}^{n} A^{-1}_{k\,k'}\,b_{k'} \sum_{j'=1}^{n} A^{-1}_{j\,j'}\,b_{j'}\right) A^{-1}_{m\,p}$$

$$+ A^{-1}_{k\,m}\,A^{-1}_{j\,p} + A^{-1}_{k\,p}\,A^{-1}_{j\,m} \Bigg\} e^{w_2(\boldsymbol{b})}$$

$$\begin{aligned} \rightarrow \quad \langle x_p\,x_m\,x_j\,x_k\rangle &= A^{-1}_{k\,j}\,A^{-1}_{m\,p} + A^{-1}_{k\,m}\,A^{-1}_{j\,p} + A^{-1}_{k\,p}\,A^{-1}_{j\,m} \\ &= \langle x_k\,x_j\rangle\langle x_m\,x_p\rangle + \langle x_k\,x_m\rangle\langle x_j\,x_p\rangle + \langle x_k\,x_p\rangle\langle x_j\,x_m\rangle \\ &= \sum_{\{a,b,c,d\}} A^{-1}_{a\,b}\,A^{-1}_{c\,d} \\ &= \sum_{\{a,b,c,d\}} \langle x_a\,x_b\rangle\langle x_c\,x_d\rangle \end{aligned}$$

where $\{a, b, c, d\}$ = All possible pairings of $\{p\ m\ j\ k\}$.

Continuing the process, we see that

$$\langle x_{k_1} \ldots x_{k_m}\rangle = 0 \qquad \text{if} \quad m = \text{odd}$$

$$\begin{aligned} \langle x_{k_1} \ldots x_{k_m}\rangle &= \sum_{\{a_1,b_1\ldots a_{m/2},b_{m/2}\}} A^{-1}_{a_1\,b_1} \ldots A^{-1}_{a_{m/2}\,b_{m/2}} \qquad \text{if} \quad m = \text{even} \\ &= \sum_{\{a_1,b_1\ldots a_{m/2},b_{m/2}\}} \langle x_{a_1}\,x_{b_1}\rangle \ldots \langle x_{a_{m/2}}\,x_{b_{m/2}}\rangle \end{aligned}$$

where

$\{a_1, b_1 \ldots a_{m/2}, b_{m/2}\}$ = All possible pairings of $\{k_1 \ldots k_m\}$.

which is just the Wick's theorem.