1.2. Perturbation Theory, Connected Contributions, Steepest Descent

1.2.1. Perturbation Theory

Let

$$\mathcal{Z}(\lambda) = \int d^n x\, e^{-A_2(\boldsymbol{x}) - \lambda V(\boldsymbol{x})} \qquad (1.15)$$

$$= \mathcal{Z}(0) \left\langle e^{-\lambda V(\boldsymbol{x})} \right\rangle$$

where we've used the generalized form of eq(1.9)

$$\langle F(\boldsymbol{x}) \rangle \equiv \frac{1}{\mathcal{Z}(0)} \int d^n x\, e^{-A_2(\boldsymbol{x})} F(\boldsymbol{x})$$

$$e^{-\lambda V(\boldsymbol{x})} = \sum_{k=0}^{\infty} \frac{(-\lambda V)^k}{k!}$$

$$\rightarrow \qquad \mathcal{Z}(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int d^n x\, e^{-A_2(\boldsymbol{x})} V^k(\boldsymbol{x})$$

$$= \mathcal{Z}(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(\boldsymbol{x}) \right\rangle \qquad (1.16)$$

Setting $F(\boldsymbol{x}) = e^{-\lambda V(\boldsymbol{x})}$ to eq(1.7)

$$\langle F(\boldsymbol{x}) \rangle = F\left(\frac{\partial}{\partial b}\right) e^{w_2(\boldsymbol{b})} \Big|_{\boldsymbol{b}=0}$$

we have

$$\mathcal{Z}(\lambda) = \mathcal{Z}(0) \left\langle e^{-\lambda V(\boldsymbol{x})} \right\rangle$$

$$= \mathcal{Z}(0) \exp\left[-\lambda V\left(\frac{\partial}{\partial b}\right)\right] e^{w_2(\boldsymbol{b})} \Big|_{\boldsymbol{b}=0}$$

1.2.2. Connected Contributions

In applying the Wick's theorem to $\left\langle V^k(\boldsymbol{x}) \right\rangle$, we define the m^{th} order connected contribution

$\left\langle V^m(\boldsymbol{x}) \right\rangle_c \equiv$ sum of all product pairing terms that involve ALL m factors of V

For example, setting $V_j = V(\boldsymbol{x}_j)$, we have

$$\langle V(\boldsymbol{x}) \rangle_c = \langle V(\boldsymbol{x}) \rangle$$

$$\left\langle V^2(\boldsymbol{x}) \right\rangle_c = \langle V_1 V_2 \rangle$$

$$\left\langle V^4(\boldsymbol{x}) \right\rangle_c = \langle V_1 V_2 \rangle \langle V_3 V_4 \rangle + \langle V_1 V_3 \rangle \langle V_2 V_4 \rangle + \langle V_1 V_4 \rangle \langle V_2 V_3 \rangle$$

The terms in $\left\langle V^k(\boldsymbol{x}) \right\rangle$ can then be grouped into terms of the form

$$\left\langle V^{k_1}(\boldsymbol{x}) \right\rangle_c^{\alpha_1} \ldots \left\langle V^{k_m}(\boldsymbol{x}) \right\rangle_c^{\alpha_m} \qquad \text{with} \qquad k = \alpha_1 k_1 + \ldots + \alpha_m k_m$$

where all k, α are non-negative integers and k_j appears α_j times in the decomposition of k.

For a given $k = \alpha_1 k_1 + \ldots + \alpha_m k_m$, there're $\dfrac{k!}{(k_1!)^{\alpha_1} \ldots (k_m!)^{\alpha_m} \alpha_1! \ldots \alpha_m!}$ distinct $\left\langle V^{k_1}(\boldsymbol{x}) \right\rangle_c^{\alpha_1} \ldots \left\langle V^{k_m}(\boldsymbol{x}) \right\rangle_c^{\alpha_m}$ terms which all evaluate to the same value. Hence, we have

$$\left\langle V^k(\boldsymbol{x}) \right\rangle = \sum_{\substack{k_1,..,k_m \\ \alpha_1,...,\alpha_m}}^{(k)} \frac{k!}{(k_1!)^{\alpha_1} \ldots (k_m!)^{\alpha_m}} \frac{\left\langle V^{k_1}(\boldsymbol{x})\right\rangle_c^{\alpha_1}}{\alpha_1!} \ldots \frac{\left\langle V^{k_m}(\boldsymbol{x})\right\rangle_c}{\alpha_m!}$$

where the sum is over all distinct integer solutions to $k = \alpha_1 k_1 + \ldots + \alpha_m k_m$.

$$\therefore \qquad \mathcal{Z}(\lambda) = \mathcal{Z}(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(\boldsymbol{x}) \right\rangle$$

$$= \mathcal{Z}(0) \sum_{k=0}^{\infty} \sum_{\substack{k_1,..,k_m \\ \alpha_1,...,\alpha_m}}^{(k)} \frac{(-\lambda)^k}{(k_1!)^{\alpha_1} \ldots (k_m!)^{\alpha_m}} \frac{\left\langle V^{k_1}(\boldsymbol{x})\right\rangle_c^{\alpha_1}}{\alpha_1!} \ldots \frac{\left\langle V^{k_m}(\boldsymbol{x})\right\rangle_c}{\alpha_m!}$$

$$= \mathcal{Z}(0) \sum_{\substack{k_1,..,k_m \\ \alpha_1,...,\alpha_m}} \frac{(-\lambda)^{\alpha_1 k_1 + \ldots + \alpha_m k_m}}{(k_1!)^{\alpha_1} \ldots (k_m!)^{\alpha_m}} \frac{\left\langle V^{k_1}(\boldsymbol{x})\right\rangle_c^{\alpha_1}}{\alpha_1!} \ldots \frac{\left\langle V^{k_m}(\boldsymbol{x})\right\rangle_c}{\alpha_m!}$$

$$= \mathcal{Z}(0) \prod_{i=0}^{\infty} \sum_{\alpha=0}^{\infty} \left[\frac{(-\lambda)^{k_i}}{k_i!} \left\langle V^{k_i}(\boldsymbol{x})\right\rangle_c \right]^{\alpha} \frac{1}{\alpha!}$$

$$= \mathcal{Z}(0) \prod_{i=0}^{\infty} \exp\left[\frac{(-\lambda)^{k_i}}{k_i!} \left\langle V^{k_i}(\boldsymbol{x})\right\rangle_c \right]$$

$$\therefore \qquad \mathcal{W}(\lambda) \equiv \ln \mathcal{Z}(\lambda) = \ln \mathcal{Z}(0) + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(\boldsymbol{x}) \right\rangle_c$$

The factor $\dfrac{k!}{(k_1!)^{\alpha_1} \ldots (k_m!)^{\alpha_m} \alpha_1! \ldots \alpha_m!}$ is obtained as follows.

1. Permutations of coordinates within each $\left\langle V^{k_j}(\boldsymbol{x}) \right\rangle = \left\langle V_1 \ldots V_{k_j} \right\rangle$ term give $k_j!$ equivalent terms.
2. Hence, permutations within each V^{k_j} factor in
   $$\left\langle V^{k_j}(\boldsymbol{x}) \right\rangle^{\alpha_j} = \left\langle V_1 \ldots V_{k_j} \right\rangle \ldots \left\langle V_{(\alpha_j-1)k_j} \ldots V_{\alpha_j k_j} \right\rangle$$
   give $\left(k_j!\right)^{\alpha_j}$ equivalent terms.
3. Finally, permutations among the α_j factors of V^{k_j} gives $\alpha_j!$ equivalent terms.

For example, setting

$$\langle i\, j\, \ldots \rangle = \left\langle V(\boldsymbol{x}_i)\, V\left(\boldsymbol{x}_j\right) \ldots \right\rangle_c$$

we have

$$\begin{aligned}\langle V(\boldsymbol{x})\rangle_c \left\langle V^2(\boldsymbol{x})\right\rangle_c^2 = &\langle 1\rangle\langle 23\rangle\langle 45\rangle + \langle 1\rangle\langle 24\rangle\langle 35\rangle + \langle 1\rangle\langle 25\rangle\langle 34\rangle \\ &+\langle 2\rangle\langle 13\rangle\langle 45\rangle + \langle 2\rangle\langle 14\rangle\langle 35\rangle + \langle 2\rangle\langle 15\rangle\langle 34\rangle \\ &+\langle 3\rangle\langle 12\rangle\langle 45\rangle + \langle 3\rangle\langle 15\rangle\langle 24\rangle + \langle 3\rangle\langle 14\rangle\langle 25\rangle \\ &+\langle 4\rangle\langle 15\rangle\langle 23\rangle + \langle 4\rangle\langle 12\rangle\langle 35\rangle + \langle 4\rangle\langle 13\rangle\langle 25\rangle \\ &+\langle 5\rangle\langle 14\rangle\langle 23\rangle + \langle 5\rangle\langle 13\rangle\langle 24\rangle + \langle 5\rangle\langle 12\rangle\langle 34\rangle\end{aligned}$$

which contains $\dfrac{5!}{1!\,(2!)^2 \cdot 1!\, 2!} = 15$ inequivalent terms.

For each inequivalent term, there're 7 other equivalent ones that can be obtained by permutation. For example,

$$\begin{aligned}&\langle 23\rangle\langle 45\rangle = \langle 32\rangle\langle 45\rangle = \langle 23\rangle\langle 54\rangle = \langle 32\rangle\langle 54\rangle \\ &= \langle 45\rangle\langle 23\rangle = \langle 45\rangle\langle 32\rangle = \langle 54\rangle\langle 23\rangle = \langle 54\rangle\langle 32\rangle\end{aligned}$$

1.2.3. Steepest Descent

Consider

$$\mathcal{I}(\lambda) = \int d^n x \, e^{-A(\mathbf{x})/\lambda}$$

Let

$$\left.\frac{\partial}{\partial x_i} A(\mathbf{x})\right|_{\mathbf{x}=\mathbf{x}^c} = 0 \qquad \forall\, i$$

&

$$\mathbf{x} = \mathbf{x}^c + \sqrt{\lambda}\, \mathbf{y}$$

$$\rightarrow \quad \frac{1}{\lambda} A(\mathbf{x}) = \frac{1}{\lambda} A(\mathbf{x}^c) + \frac{1}{2} \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j} y_i y_j + \sum_{k=3}^{\infty} \frac{\lambda^{k/2-1}}{k!} \frac{\partial^k A(\mathbf{x}^c)}{\partial x_{i_1} \ldots \partial x_{i_k}} y_{i_1} \ldots y_{i_k}$$

(summation over repeated indices understood)

$$\therefore \quad \mathcal{I}(\lambda) = \lambda^{n/2} e^{-A(\mathbf{x}^c)/\lambda} \int d^n y \, e^{-A_2(\mathbf{y}) - R(\mathbf{y})}$$

where

$$A_2(\mathbf{y}) = \frac{1}{2} \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j} y_i y_j$$

$$R(\mathbf{y}) = \sum_{k=3}^{\infty} \frac{\lambda^{k/2-1}}{k!} \frac{\partial^k A(\mathbf{x}^c)}{\partial x_{i_1} \ldots \partial x_{i_k}} y_{i_1} \ldots y_{i_k}$$

For $\lambda \rightarrow 0$,

$$\mathcal{I}(\lambda) \approx \lambda^{n/2} e^{-A(\mathbf{x}^c)/\lambda} \int d^n y \, e^{-A_2(\mathbf{y})}$$

$$= \lambda^{n/2} e^{-A(\mathbf{x}^c)/\lambda} \sqrt{\frac{(2\pi)^n}{\det \dfrac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j}}}$$