1.3. Complex Structures

Consider

$$Z(\boldsymbol{A};\ \boldsymbol{b},\ \boldsymbol{c}) \equiv \int d^n x \int d^n y\, e^{-A_2(\boldsymbol{x})-A_2(\boldsymbol{y})+\boldsymbol{b}^T\boldsymbol{x}+\boldsymbol{c}^T\boldsymbol{y}}$$

$$= \int d^n x\, e^{-A_2(\boldsymbol{x})+\boldsymbol{b}^T\boldsymbol{x}} \int d^n y\, e^{-A_2(\boldsymbol{y})+\boldsymbol{c}^T\boldsymbol{y}}$$

where

$$A_2(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A}\,\boldsymbol{x} = \frac{1}{2} x_i A_{ij} x_j$$

and $\boldsymbol{A}$ is real, symmetric & positive definite.

$$\mathcal{Z}(\boldsymbol{A},\boldsymbol{b}) = \int d^n x\, e^{-A_2(\boldsymbol{x})+\boldsymbol{b}^T\boldsymbol{x}} = \frac{(2\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}} \exp\left(\frac{1}{2}\boldsymbol{b}^T\boldsymbol{A}^{-1}\boldsymbol{b}\right)$$

$$\to \qquad Z(\boldsymbol{A};\boldsymbol{b},\boldsymbol{c}) = \frac{(2\pi)^n}{\det \boldsymbol{A}} \exp\left(\frac{1}{2}\boldsymbol{b}^T\boldsymbol{A}^{-1}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{c}\right)$$

Let

$$z_i = \frac{1}{\sqrt{2}}(x_i + i\,y_i) \qquad\qquad z_i^* = \frac{1}{\sqrt{2}}(x_i - i\,y_i) \qquad\qquad (1.26)$$

$$\to \qquad x_i = \frac{1}{\sqrt{2}}(z_i + z_i^*) \qquad\qquad y_i = \frac{1}{i\sqrt{2}}(z_i - z_i^*)$$

$$\to \qquad \frac{\partial(x_i, y_i)}{\partial(z_i^*, z_i)} = \begin{vmatrix} \frac{\partial x_i}{\partial z_i^*} & \frac{\partial x_i}{\partial z_i} \\ \frac{\partial y_i}{\partial z_i^*} & \frac{\partial y_i}{\partial z_i} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{i\sqrt{2}} & \frac{1}{i\sqrt{2}} \end{vmatrix} = -i$$

$$\therefore \qquad \int d\,x_i \int d\,y_i = \int d\,z_i^* \int d\,z_i \frac{\partial(x_i, y_i)}{\partial(z_i^*, z_i)} = -i \int d\,z_i^* \int d\,z_i \qquad\qquad (a)$$

Note: The sign ambiguity of the jacobian can be removed using notations of differential form. Thus, eq(a) really means

$$\int d\,x_i \wedge d\,y_i = -i \int d\,z_i^* \wedge d\,z_i$$

$$z_i^* A_{ij} z_j = \frac{1}{2}(x_i - i\,y_i) A_{ij} (x_j + i\,y_j)$$

$$= \frac{1}{2}(x_i A_{ij} x_j + y_i A_{ij} y_j) \qquad\qquad \text{since} \quad A_{ij} = A_{ji}$$

$$= A_2(\boldsymbol{x}) + A_2(\boldsymbol{y})$$

$$= \boldsymbol{z}^+ \boldsymbol{A}\,\boldsymbol{z}$$

Let $\quad a = \frac{1}{\sqrt{2}}(s + i\,t),$

$$\to \qquad a^* z + a z^* = \frac{1}{2}[(s - i\,t)(x + i\,y) + (s + i\,t)(x + i\,y)]$$

$$= s\,x + t\,y$$

Let $\quad \boldsymbol{a} = \frac{1}{\sqrt{2}}(\boldsymbol{b} + i\,\boldsymbol{c}) \qquad\qquad a_i = \frac{1}{\sqrt{2}}(b_i + i\,c_i)$

$$\to \qquad \boldsymbol{a}^+ \boldsymbol{z} + \boldsymbol{z}^+ \boldsymbol{a} = a_i^* z_i + z_i^* a_i = \boldsymbol{b}^T \boldsymbol{x} + \boldsymbol{c}^T \boldsymbol{y}$$

$$\therefore \qquad Z(\boldsymbol{A};\ \boldsymbol{b},\ \boldsymbol{c}) = \int \prod_{i=1}^{n} \frac{d z_i^* \, d z_i}{i} \ e^{-\boldsymbol{z}^+ \boldsymbol{A} \boldsymbol{z} + \boldsymbol{a}^+ \boldsymbol{z} + \boldsymbol{z}^+ \boldsymbol{a}}$$

$$= \frac{(2\pi)^n}{\det \boldsymbol{A}} \exp\left(\frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b} + \frac{1}{2} \boldsymbol{c}^T \boldsymbol{A}^{-1} \boldsymbol{c}\right)$$

$$= \frac{(2\pi)^n}{\det \boldsymbol{A}} \exp\left(\boldsymbol{a}^+ \boldsymbol{A}^{-1} \boldsymbol{a}\right)$$

$$\rightarrow \qquad \mathcal{Z}(\boldsymbol{A};\ \boldsymbol{a},\ \boldsymbol{a}^+) \equiv \int \prod_{i=1}^{n} \frac{d z_i^* \, d z_i}{2\pi i} \ e^{-\boldsymbol{z}^+ \boldsymbol{A} \boldsymbol{z} + \boldsymbol{a}^+ \boldsymbol{z} + \boldsymbol{z}^+ \boldsymbol{a}} \qquad (1.27)$$

$$= \frac{1}{\det \boldsymbol{A}} \exp\left(\boldsymbol{a}^+ \boldsymbol{A}^{-1} \boldsymbol{a}\right) \qquad (1.29)$$

$$= \frac{1}{\det \boldsymbol{A}} \exp\left(a_i^* A_{ij}^{-1} a_j\right)$$

$$= \frac{1}{\det \boldsymbol{A}} e^{u(\boldsymbol{a})}$$

where $\qquad u(\boldsymbol{a}) = \boldsymbol{a}^+ \boldsymbol{A}^{-1} \boldsymbol{a} = a_i^* A_{ij}^{-1} a_j$

$$\therefore \qquad \langle z_i \ \ldots \ z_k^* \ \ldots \rangle = \frac{1}{\mathcal{Z}(\boldsymbol{A})} \frac{\partial}{\partial a_i^*} \ldots \frac{\partial}{\partial a_k} \ldots \mathcal{Z}(\boldsymbol{A};\ \boldsymbol{a},\ \boldsymbol{a}^+)$$

$$= \frac{\partial}{\partial a_i^*} \ldots \frac{\partial}{\partial a_k} \ldots e^{u(\boldsymbol{a})} \Big|_{\boldsymbol{a}=0}$$

Using

$$\frac{\partial}{\partial a_k} e^{u(\boldsymbol{a})} = \sum_{i=1}^{n} a_i^* A_{ik}^{-1} e^{u(\boldsymbol{a})} \qquad \qquad \frac{\partial}{\partial a_k^*} e^{u(\boldsymbol{a})} = \sum_{j=1}^{n} A_{kj}^{-1} a_j e^{u(\boldsymbol{a})}$$

to follow the procedure in §1.1, we have

$$\left\langle z_{k_1},\ \ldots,\ z_{k_m},\ z_{j_1}^*,\ \ldots,\ z_{j_p}^* \right\rangle = 0 \qquad \text{if} \quad m \neq p$$

$$\left\langle z_{k_1},\ \ldots,\ z_{k_m},\ z_{j_1}^*,\ \ldots,\ z_{j_m}^* \right\rangle = \sum_{\{a_1 b_1, \ldots, a_m, b_m\}} A_{a_1 b_1}^{-1} \ldots A_{a_m b_m}^{-1}$$

$$= \sum_{\{a_1 b_1, \ldots, a_m, b_m\}} \left\langle z_{a_1} z_{b_1}^* \right\rangle \ldots \left\langle z_{a_m} z_{b_m}^* \right\rangle$$

where

$\{a_1,\ b_1 \ldots a_m,\ b_m\} =$ All possible pairings of $\{k_1 \ldots k_m,\ j_1,\ \ldots j_m\}$
with $a_i \in \{k_1 \ldots k_m\}$ & $b_i \in \{j_1,\ \ldots j_m\}$