1.4. Grassmann Algebras, Differential Forms

Grassmann Algebra

A Grassmann (exterior) algebra $\mathfrak{U}$ over $\mathbb{R}$ or $\mathbb{C}$ is an associative algebra constructed from $\{1, \theta_1, \ldots, \theta_n\}$, where the generators θ_i are anti-symmetric, i.e.,

$$\theta_i \theta_j = -\theta_j \theta_i \qquad \forall\ i, j \tag{1.31}$$

$\rightarrow \qquad \theta_i^2 = -\theta_i^2 = 0$

Thus,

1. All elements of $\mathfrak{U}$ are polynomials of at most 1st degree in all θ_i.
2. The basis of $\mathfrak{U}$ is the set
$$\left\{ \theta_1^{j_1} \ldots \theta_n^{j_n} \;\middle|\; j_k \in \{0, 1\} \right\}$$
where $\theta_j^0 = 1$ is understood.
3. Dimension of $\mathfrak{U}$ is therefore 2^n.

For example, $n = 2$ gives $\{1, \theta_1, \theta_2\}$ so that the basis is

$$\{ \theta_1^0 \theta_2^0,\ \theta_1^1 \theta_2^0,\ \theta_1^0 \theta_2^1,\ \theta_1^1 \theta_2^1 \} = \{1, \theta_1, \theta_2, \theta_1 \theta_2\}$$

The dimension of $\mathfrak{U}$ is indeed $4 = 2^2$.

The degree of a monomial $\theta_1^{j_1} \ldots \theta_m^{j_m}$ is $\sum_{k=1}^{m} j_k$.

Monomials with the highest degree are therefore multiples of $\theta_1 \ldots \theta_n$.

Polynomials formed by linear combinations of monomials of the same degree are called homogeneous.

Obviously, all homogeneous polynomials of the same degree form a linear vector space. The vector space $\mathfrak{U}$ is therefore partitioned into subspaces of different degrees (or grades). In this sense, $\mathfrak{U}$ is a graded algebra.

Note that an element of $\mathfrak{U}$ is invertible if and only if it contains the basis 1.

For example, element $1 + \theta$ is invertible.
Its inverse is $1 - \theta$ since

$$(1 + \theta)(1 - \theta) = 1 = (1 - \theta)(1 + \theta)$$

Grassmannian Parity

Parity operator P is defined by

$$P(\theta_i) = -\theta_i \qquad \forall\ i \tag{1.32}$$

$\rightarrow \qquad P^2(\theta_i) = P(-\theta_i) = \theta_i$

$\therefore \qquad P^2 = 1 \qquad$ (idempotent)

$$\begin{aligned} P(\theta_{i_1} \ldots \theta_{i_p}) &= P(\theta_{i_1}) \ldots P(\theta_{i_p}) \\ &= (-\theta_{i_1}) \ldots (-\theta_{i_p}) \\ &= (-)^p\, \theta_{i_1} \ldots \theta_{i_p} \end{aligned} \tag{1.33}$$

i.e., monomials of even(odd) degree in θ_i has even(odd) parity.

Let $A^{(m)}$ & $B^{(n)}$ be homogeneous polynomials of degrees m & n, respectively, then

$$A^{(m)} B^{(n)} = (-)^{mn} B^{(n)} A^{(m)}$$

$$= \begin{cases} -B^{(n)} A^{(m)} & \text{if both } n \text{ \& } m \text{ are odd} \\ B^{(n)} A^{(m)} & \text{otherwise} \end{cases}$$

For example,

$$(\theta_1 \theta_2)(\theta_3 \theta_4 \theta_5) = (-)^3 \theta_1 (\theta_3 \theta_4 \theta_5) \theta_2$$
$$= (-)^6 (\theta_3 \theta_4 \theta_5)(\theta_1 \theta_2)$$
$$= (\theta_3 \theta_4 \theta_5)(\theta_1 \theta_2)$$

Let

$$P(\mathfrak{U}^\pm) = \pm \mathfrak{U}^\pm \tag{1.34}$$

→ $\mathfrak{U}^+$ is a (commuting) sub-algebra of $\mathfrak{U}$ & is spanned by $\theta_{i_1} \ldots \theta_{i_p}$ of even p.
$\mathfrak{U}^-$ does not form an algebra since 1 is not in it.

Since every element of the basis $\left\{ \theta_1^{j_1} \ldots \theta_n^{j_n} \,\middle|\, j_k \in \{0, 1\} \right\}$ must belong either to $\mathfrak{U}^+$ or $\mathfrak{U}^-$, any $A \in \mathfrak{U}$ can be written as

$$A = A^+ + A^- \qquad \text{with} \qquad A^\pm \in \mathfrak{U}^\pm$$

This means any element of $\mathfrak{U}^+$ commutes with every element of $\mathfrak{U}$.
Elements of $\mathfrak{U}^-$ anti-commute with each other but commute with elements of $\mathfrak{U}^+$.

Differential Forms

Let $\Omega_{\mu_1 \ldots \mu_m}(x)$ be a totally antisymmetric tensor. Then

$$\Omega = \Omega_{\mu_1 \ldots \mu_m}(x)\, \theta^{\mu_1} \ldots \theta^{\mu_m} \tag{1.35}$$

is an m-form (with $\theta^\mu \theta^\nu$ taking the place of $d x^\mu \wedge d x^\nu$).

For $\mathfrak{U}$ of n generators, any m-form vanishes if $m > n$.

The differential operator is defined by

$$d \equiv \theta^\mu \frac{\partial}{\partial x^\mu} \tag{1.36}$$

Since $\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu$ & $\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu}$, we have

$$d^2 = \theta^\mu \frac{\partial}{\partial x^\mu} \theta^\nu \frac{\partial}{\partial x^\nu}$$
$$= \theta^\mu \theta^\nu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -\theta^\nu \theta^\mu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$$
$$= \theta^\nu \theta^\mu \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} = \theta^\nu \theta^\mu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu}$$
$$= 0 \tag{1.37}$$

In differential geometry,

$d\,\Omega = 0 \Leftrightarrow$ Ω is closed
$\Omega = d\,\Omega'$ $\Leftrightarrow$ Ω is exact

Since

$$d\,\Omega = d^2\, \Omega' = 0$$

∴ All exact forms are closed.