1.5. Differentiation in Grassmann Algebras

Consider any $A \in \mathfrak{U}$ as a function of θ_i, we can write

$$A = A_1 + \theta_i A_2 = A_1 + A_3 \theta_i$$

where A_1, A_2 & A_3 are independent of θ_i but may be functions of $\{ \theta_j \mid j \neq i \}$.

The (left-) differentiation operator is defined as

$$\begin{aligned} \frac{\partial}{\partial \theta_i} A &= \lim_{\Delta\theta_i \to 0} \frac{1}{\Delta\theta_i} [A(\theta_i + \Delta\theta_i) - A(\theta_i)] \\ &= \lim_{\Delta\theta_i \to 0} \frac{1}{\Delta\theta_i} (\Delta\theta_i A_2) \\ &= A_2 \end{aligned} \qquad (1.38)$$

with

$$\left(\frac{\partial}{\partial \theta_i} \right)^2 A = 0$$

The (right-) differentiation operator is defined as

$$\begin{aligned} A \frac{\overleftarrow{\partial}}{\partial \theta_i} &= \lim_{\Delta\theta_i \to 0} [A(\theta_i + \Delta\theta_i) - A(\theta_i)] \frac{1}{\Delta\theta_i} \\ &= \lim_{\Delta\theta_i \to 0} (A_3 \Delta\theta_i) \frac{1}{\Delta\theta_i} = A_3 \end{aligned}$$

with

$$A \left(\frac{\overleftarrow{\partial}}{\partial \theta_i} \right)^2 = 0$$

Chain Rule

Let $\quad f(\theta) = u[v(\theta)]$

$\rightarrow \quad \frac{\partial}{\partial\theta} f(\theta) = \lim_{\Delta\theta \to 0} \frac{1}{\Delta\theta} \{ u[v(\theta + \Delta\theta)] - u[v(\theta)] \}$

Setting

$$v(\theta + \Delta\theta) = v(\theta) + \Delta v$$

we have

$$\begin{aligned} \frac{\partial}{\partial\theta} f(\theta) &= \lim_{\Delta\theta,\, \Delta v \to 0} \frac{1}{\Delta\theta} \Delta v \left\{ \frac{1}{\Delta v} [u(v + \Delta v) - u(v)] \right\} \\ &= \frac{\partial v}{\partial\theta} \frac{\partial u}{\partial v} \\ &\equiv \frac{\partial v}{\partial\theta} \frac{\partial f}{\partial v} \end{aligned} \qquad \text{(Chain rule)}$$

Note that in general,

$$\frac{\partial v}{\partial\theta} \frac{\partial f}{\partial v} \neq \frac{\partial f}{\partial v} \frac{\partial v}{\partial\theta}$$

Let

$$\sigma(\theta) \in \mathfrak{U}^- \qquad\qquad x(\theta) \in \mathfrak{U}^+$$

$\rightarrow$

$$\frac{\partial}{\partial\theta} f(\sigma, x) = \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} \qquad (1.39)$$

Since $\frac{\partial}{\partial\theta}$ reduces the degree of every monomial by 1, we have

$$\frac{\partial\sigma}{\partial\theta} \in \mathfrak{U}^+ \qquad\qquad \frac{\partial x}{\partial\theta} \in \mathfrak{U}^-$$

$$\rightarrow \quad \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} = \frac{\partial f}{\partial\sigma}\frac{\partial\sigma}{\partial\theta}$$

but

$$\frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} \neq \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}$$

(See sub-section “Grassmannian Parity” of §1 .4. Grassmann Algebras Differential Forms)

Formal Construction

A Grassmann differential operator (or anti-derivation) D on $\mathfrak{U}$

1. is a linear mapping into itself, i.e.,

$$D(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 D A_1 + \lambda_2 D A_2 \qquad (1.40)$$
$$\forall\ \lambda_1, \lambda_2 \in \mathbb{R} \text{ or } \mathbb{C}$$

2. obeys the anti-derivation Leibniz rule, i.e.,

$$D(A_1 A_2) = D(A_1) A_2 + P(A_1) D(A_2) \qquad (1.41)$$

$$\rightarrow \quad D(\theta_i \theta_j) = D(\theta_i)\theta_j - \theta_i D(\theta_j)$$

Property (2.) follows from the fact that D lowers the degree of every monomial by one. Thus, if $D A \neq 0$, the parity of $D A$ is opposite to that of A, i.e.,

$$D A^{\pm} \in \mathfrak{U}^{\mp} \qquad \text{where} \qquad A^{\pm} \in \mathfrak{U}^{\pm}$$

More generally,

$$A = A^+ + A^- \qquad\qquad \forall\ A \in \mathfrak{U}$$
$$\rightarrow \quad P A = A^+ - A^-$$
$$\therefore \quad D P A = D A^+ - D A^-$$

$$D A = D A^+ + D A^-$$
$$\rightarrow \quad P D A = -D P A^+ - D P A^-$$
$$= -D A^+ + D A^-$$
$$= -D P A$$

i.e.,

$$D P = -P D \qquad (1.42)$$

Eqs (1.41) & (1.42) are equivalent. Thus, with $P^2 = 1$, we have

$$D P(A_1 A_2) = D[P(A_1) P(A_2)]$$
$$= D(P A_1) P(A_2) + A_1 D(P A_2)$$
$$P D(A_1 A_2) = P[D(A_1) A_2 + P(A_1) D(A_2)]$$
$$= P(D A_1) P(A_2) + A_1 P D(A_2)$$
$$= -D(P A_1) P(A_2) - A_1 D(P A_2)$$
$$= -D P(A_1 A_2)$$

as required.

$$D(A^2) = D(A) A + P(A) D(A)$$

If $A \in \mathfrak{U}^+$, then

$$D(A^2) = D(A) A + A D(A) \;= 2 D(A) A \;= 2 A D(A)$$

Similarly, one can show that

$$D\left(A^k\right) = k\ D(A)\, A^{k-1} = n\, A^{k-1}\, D(A)$$

Let $F(x)$ be any regular function, i.e.,

$$F(x) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} x^k$$

$$F'(x) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{(k-1)!} x^{k-1}$$

If $A \in \mathfrak{U}^+$, then

$$F(A) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} A^k \qquad F'(A) = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{(k-1)!} A^{k-1}$$

$$D[F(A)] = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} D\left(A^k\right)$$

$$= D(A) \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{(k-1)!} A^{k-1} = \sum_{k=1}^{\infty} \frac{F^{(k)}(0)}{(k-1)!} A^{k-1}\, D(A)$$

$$= D(A)\, F'(A) = F'(A)\, D(A) \qquad (1.43)$$

Consider the expansion of $A \in \mathfrak{U}$ in terms of the basis $\{1,\ \theta_i,\ \theta_i\,\theta_j\,,\ \ldots\}$.
If A includes the identity 1, i.e.,

$$A = a + \ldots$$

where $a \neq 0$ is a constant, then

$$A^k = a^k + \ldots \neq 0 \qquad \forall\ k$$

If A does not contain the identity 1, then every monomial term in A^k would have a degree greater or equal to k. Hence, in an algebra of n generators,

$$A^k = 0 \qquad \forall\ k > n$$

In general, let m be the smallest degree in the expansion of A, then

$$A^k = 0 \qquad \forall\ k > n/m$$

Anticommutation Relations

Let D & D' be two anti-differential operators, i.e.,

$$D(\lambda_1\, A_1 + \lambda_2\, A_2) = \lambda_1\, D\, A_1 + \lambda_2\, D\, A_2$$

$$D'(\lambda_1\, A_1 + \lambda_2\, A_2) = \lambda_1\, D'\, A_1 + \lambda_2\, D'\, A_2$$

&

$$D(A_1\, A_2) = D(A_1)\, A_2 + P(A_1)\, D(A_2)$$

$$D'(A_1\, A_2) = D'(A_1)\, A_2 + P(A_1)\, D'(A_2)$$

$\rightarrow$

$$D\,D'(\lambda_1\, A_1 + \lambda_2\, A_2) = D\,(\lambda_1\, D'\, A_1 + \lambda_2\, D'\, A_2)$$

$$= \lambda_1\, D\,D'\, A_1 + \lambda_2\, D\,D'\, A_2$$

$$D'\,D(\lambda_1\, A_1 + \lambda_2\, A_2) = D'\,(\lambda_1\, D\, A_1 + \lambda_2\, D\, A_2)$$

$$= \lambda_1\, D'\,D\, A_1 + \lambda_2\, D'\,D\, A_2$$

$\therefore$

$$(D\,D' + D'\,D)\,(\lambda_1\, A_1 + \lambda_2\, A_2) = \lambda_1 (D\,D' + D'\,D)\, A_1 + \lambda_2 (D\,D' + D'\,D)\, A_2$$

Also, with $P^2 = 1$, we have

$$D\,D'(A_1\, A_2) = D[\,D'(A_1)\, A_2 + P(A_1)\, D'(A_2)\,]$$

$$= D\,D'(A_1)\, A_2 + P[D'(A_1)]\, D(A_2) + D[P(A_1)\,]\, D'(A_2) + A_1\, D\,D'(A_2)$$

$$D'\,D(A_1\, A_2) = D'\,[\,D(A_1)\, A_2 + P(A_1)\, D(A_2)\,]$$

$$= D'\,D(A_1)\, A_2 + P[D(A_1)]\, D'(A_2) + D'[P(A_1)\,]\, D(A_2) + A_1\, D'\,D(A_2)$$

Using

$$P\,D = -D\,P \qquad \& \qquad P\,D' = -D'\,P$$

we have

$$(D\,D' + D'\,D)\,(A_1\,A_2) = (D\,D' + D'\,D)\,(A_1)\,A_2 + A_1 (D\,D' + D'\,D)\,(A_2)$$

Let

$$\Delta = D\,D' + D'\,D \qquad (1.44)$$

then

$$\Delta(\lambda_1\,A_1 + \lambda_2\,A_2) = \lambda_1\,\Delta\,A_1 + \lambda_2\,\Delta\,A_2$$
$$\Delta(A_1\,A_2) = \Delta(A_1)\,A_2 + A_1\,\Delta(A_2) \qquad (1.45)$$

i.e., Δ is the usual differential operator satisfying the Leibniz rule.

Furthermore,

$$\begin{aligned} P\,\Delta &= P(D\,D' + D'\,D) \\ &= -D\,P\,D' - D'\,P\,D \\ &= D\,D'\,P + D'\,D\,P \\ &= (D\,D' + D'\,D)\,P \\ &= \Delta\,P \end{aligned} \qquad (1.46)$$

A Basis

D is defined by its action on $A \in \mathfrak{U}$ through eqs(1.40-1).

Also, from the proof of eq(1.42), we see that

$$B\,D\,P = -P\,B\,D \qquad \text{if } B \in \mathfrak{U}^+$$

which means $B\,D$ is also an anti-derivation operator.

Any $A \in \mathfrak{U}$ is a linear combinations of the basis vectors obtained from the generators $\{\theta_i\}$. This means any D can be expanded in terms of a basis defined by its action on $\{\theta_i\}$, with elements of $\mathfrak{U}^+$ as its left coefficients of combination.

The simplest such basis is $\left\{\frac{\partial}{\partial\,\theta_i}\right\}$ defined by

$$\frac{\partial}{\partial\,\theta_i}\,\theta_j = \delta_{i\,j} \qquad (1.47)$$

Treating $\{\theta_i\}$ also as operators on $\mathfrak{U}$, the algebra of operators on $\mathfrak{U}$ obeys anti-commutation rules as follows

1.) $\quad \theta_i\,\theta_j + \theta_j\,\theta_i = 0 \qquad (1.31 / 1.48a)$

2.) $$\frac{\partial}{\partial\,\theta_i}\,\frac{\partial}{\partial\,\theta_j}(\theta_k\,\theta_l) = \frac{\partial}{\partial\,\theta_i}\left(\delta_{j\,k}\,\theta_l - \theta_k\,\delta_{j\,l}\right) = \delta_{j\,k}\,\delta_{i\,l} - \delta_{i\,k}\,\delta_{j\,l}$$

$$\frac{\partial}{\partial\,\theta_j}\,\frac{\partial}{\partial\,\theta_i}(\theta_k\,\theta_l) = \frac{\partial}{\partial\,\theta_j}\left(\delta_{i\,k}\,\theta_l - \theta_k\,\delta_{i\,l}\right) = \delta_{i\,k}\,\delta_{j\,l} - \delta_{j\,k}\,\delta_{i\,l}$$

$$\rightarrow \qquad \frac{\partial}{\partial\,\theta_i}\,\frac{\partial}{\partial\,\theta_j} + \frac{\partial}{\partial\,\theta_j}\,\frac{\partial}{\partial\,\theta_i} = 0 \qquad (1.48b)$$

3.) $$\frac{\partial}{\partial\,\theta_j}(\theta_i\,A) = \delta_{i\,j}\,A - \theta_i\,\frac{\partial}{\partial\,\theta_j}\,A$$

$$\rightarrow \qquad \theta_i\,\frac{\partial}{\partial\,\theta_j} + \frac{\partial}{\partial\,\theta_j}\,\theta_i = \delta_{i\,j} \qquad (1.48c)$$

Let

$$D_i^{\pm} = \frac{\partial}{\partial\,\theta_i} \pm \theta_i$$

$$\rightarrow \quad \left\{D_i^{\pm}, D_j^{\pm}\right\} = \left\{\frac{\partial}{\partial \theta_i} \pm \theta_i, \frac{\partial}{\partial \theta_j} \pm \theta_j\right\}$$

$$= \left\{\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right\} \pm \left\{\frac{\partial}{\partial \theta_i}, \theta_j\right\} \pm \left\{\theta_i, \frac{\partial}{\partial \theta_j}\right\} + \left\{\theta_i, \theta_j\right\}$$

$$= \pm \left\{\frac{\partial}{\partial \theta_i}, \theta_j\right\} \pm \left\{\theta_i, \frac{\partial}{\partial \theta_j}\right\}$$

$$= \pm 2\, \delta_{i\,j} \tag{1.49a}$$

$$\left\{D_i^{+}, D_j^{-}\right\} = \left\{\frac{\partial}{\partial \theta_i} + \theta_i, \frac{\partial}{\partial \theta_j} - \theta_j\right\}$$

$$= \left\{\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right\} - \left\{\frac{\partial}{\partial \theta_i}, \theta_j\right\} + \left\{\theta_i, \frac{\partial}{\partial \theta_j}\right\} - \left\{\theta_i, \theta_j\right\}$$

$$= -\left\{\frac{\partial}{\partial \theta_i}, \theta_j\right\} + \left\{\theta_i, \frac{\partial}{\partial \theta_j}\right\}$$

$$= 0 \tag{1.49b}$$