1.6. Integration in Grassmann Algebras

General Properties

We define the integral operator I as a linear operator on $\mathfrak{U}$ that acts like a contour integral (definite integral with no boundary terms). Thus, it has the following properties.

1. It is linear.
$$I(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 I(A_1) + \lambda_2 I(A_2) \qquad (1.51)$$
2. Without boundary terms, the integral of a total derivative vanishes.
$$I D = 0 \qquad (1.52)$$
3. Derivative of a definite integral vanishes.
$$D I = 0 \qquad (1.53)$$
4. A constant factor can be taken outside the integral.
$$D(A) = 0 \quad \Rightarrow \quad I(B A) = I(B) A \qquad (1.54)$$
5. It changes the grading like D.
$$P I + I P = 0 \qquad (1.54a)$$

For any nilpotent differential operator such that $D^2 = 0$, eqs(1.51-4a) are satisfied by $I = D$.

Since any D can be expanded terms of the basis $\left\{ \frac{\partial}{\partial \theta_i} \right\}$, we need only consider

$$I_i = \frac{\partial}{\partial \theta_i}$$

and define

$$I_i A \equiv \int d\theta_i A \equiv \frac{\partial}{\partial \theta_i} A \qquad \forall\ A \in \mathfrak{U} \qquad (1.50)$$

Change of Variables in a Grassmann Integral

Consider

$$I = \int d\theta f(\theta) \qquad (1.55)$$

& a change of variable

$$\theta = a\theta' + b \qquad (1.56)$$

Note: θ should be consider as any one of the generators $\{\theta_i\}$.

In order for the inverse relation

$$\theta' = a^{-1}\theta - a^{-1} b$$

to be valid, a^{-1} must exist, i.e., a is invertible.

If θ' is to have the same parity of θ, then

$$a \in \mathfrak{U}^{+} \qquad \& \qquad b \in \mathfrak{U}^{-}$$

Using

$$\frac{\partial}{\partial \theta} = \left(\frac{\partial}{\partial \theta} \theta' \right) \frac{\partial}{\partial \theta'}$$
$$= a^{-1} \frac{\partial}{\partial \theta'}$$

we have

$$
\begin{aligned}
I &= \int d\theta\, f(\theta) = \frac{\partial}{\partial\theta} f(\theta) \\
&= a^{-1} \frac{\partial}{\partial\theta'} f(a\theta' + b) \\
&= a^{-1} \int d\theta' f(a\theta' + b) \\
&= \int d\theta' J f(\theta')
\end{aligned}
\qquad (1.57)
$$

Hence, the jacobian J of the transformation is a^{-1}.

Note that if θ is a commuting variable, then

$$d\theta = a\, d\theta'$$

$$\to \qquad I = \int d\theta\, f(\theta) = a \int d\theta' f(\theta')$$

so that $J = \dfrac{\partial\theta}{\partial\theta'} = a$.

Generalization

The above can be generalized to the change of multiple variables

$$\theta_i = \theta_i(\{\theta_j'\}) \equiv \theta_i(\theta') \qquad \text{with} \qquad \theta_i,\ \theta_j' \in \mathfrak{U}^-$$

so that

$$d\theta_1 \ldots d\theta_n = d\theta_1' \ldots d\theta_n' J \qquad (1.58)$$

Following eq(1.57), we expect

$$J = \det \frac{\partial\theta_i'}{\partial\theta_j} \in \mathfrak{U}^+ \qquad (1.59)$$

Since the change of variables must leave the value of the integral unchanged, the two sets of variables must span the same manifold. This means the transformation must be invertible, i.e., $J \neq 0$.

The formal proof of eq(1.59) is as follows.
Generalizing eq(1.50), we define

$$
\begin{aligned}
\int d\theta_1 \ldots d\theta_n\, f(\theta) &\equiv \frac{\partial}{\partial\theta_1} \cdots \frac{\partial}{\partial\theta_n} f(\theta) \\
&= \prod_{i=1}^{n} \frac{\partial}{\partial\theta_i} f(\theta)
\end{aligned}
$$

(Note the order of the product.)

Using the chain rule, we have

$$\frac{\partial}{\partial\theta_i} = \sum_{k=1}^{n} \frac{\partial\theta_k'}{\partial\theta_i} \frac{\partial}{\partial\theta_k'}$$

$$\to \qquad \prod_{i=1}^{n} \frac{\partial}{\partial\theta_i} = \prod_{i=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial\theta_k'}{\partial\theta_i} \frac{\partial}{\partial\theta_k'} \right) \qquad (1.59a)$$

where $\dfrac{\partial\theta_k'}{\partial\theta_i} \in \mathfrak{U}^+$.

Eq(1.59a) is a product of n linear combinations of $\left\{ \dfrac{\partial}{\partial\theta_k'} \right\}$. Upon expansion, it gives a homogeneous polynomial of the n^{th} degree in $\dfrac{\partial}{\partial \theta'}$. Furthermore, if the factor $\dfrac{\partial}{\partial \theta_j'}$ arises from

$$\frac{\partial}{\partial \theta_i} = \sum_{k=1}^{n} \frac{\partial \theta_k'}{\partial \theta_i} \frac{\partial}{\partial \theta_k'}$$

then it comes with a factor $\dfrac{\partial \theta_j'}{\partial \theta_i}$.

Since $\dfrac{\partial}{\partial \theta_k'} \in \mathfrak{U}^-$, we have $\left(\dfrac{\partial}{\partial \theta_k'}\right)^2 = 0$ so that every monomial term in eq(1.59a) is proportional to $\dfrac{\partial}{\partial \theta_1'} \ldots \dfrac{\partial}{\partial \theta_n'}$. Thus, we have

$$\prod_{i=1}^{n} \frac{\partial}{\partial \theta_i} = \sum_P (-)^P \frac{\partial \theta_1'}{\partial \theta_{P(1)}} \ldots \frac{\partial \theta_n'}{\partial \theta_{P(n)}} \frac{\partial}{\partial \theta_1'} \ldots \frac{\partial}{\partial \theta_n'}$$

where P is a permutation of $\{1, \ldots, n\}$ that turns it into $\{P(1), \ldots, P(n)\}$ with a parity $(-)^P$.

$$\therefore \qquad \prod_{i=1}^{n} \frac{\partial}{\partial \theta_i} = \left(\det \frac{\partial \theta_j'}{\partial \theta_m}\right) \prod_{k=1}^{n} \frac{\partial}{\partial \theta_k'}$$

$$\rightarrow \qquad \int d\theta_1 \ldots d\theta_n \, f(\theta) = \left(\det \frac{\partial \theta_j'}{\partial \theta_m}\right) \prod_{k=1}^{n} \frac{\partial}{\partial \theta_k'} f(\theta')$$

$$= \int d\theta_1' \ldots d\theta_n' \left(\det \frac{\partial \theta_j'}{\partial \theta_m}\right) f(\theta)' \qquad \text{QED}$$

Example

Let $\quad f(\theta) = \theta_n \ldots \theta_1$.

$$\rightarrow \qquad \int d\theta_1 \ldots d\theta_n \, \theta_n \ldots \theta_1 = \frac{\partial}{\partial \theta_1} \ldots \frac{\partial}{\partial \theta_n} \theta_n \ldots \theta_1 = 1$$

Let

$$\theta_i = \sum_{j=1}^{n} M_{ij} \theta_j'$$

$$\rightarrow \qquad \theta_n \ldots \theta_1 = \left(\sum_{j_n=1}^{n} M_{n j_n} \theta_{j_n}'\right) \ldots \left(\sum_{j_1=1}^{n} M_{1 j_1} \theta_{j_1}'\right)$$

$$= \sum_P (-)^P M_{n P(n)} \ldots M_{1 P(1)} \theta_n' \ldots \theta_1'$$

$$= (\det \boldsymbol{M})\, \theta_n' \ldots \theta_1'$$

Now,

$$\theta_i = \sum_{j=1}^{n} M_{ij} \theta_j'$$

$$\rightarrow \qquad \frac{\partial}{\partial \theta_k} \theta_i = \delta_{ik} = \sum_{j=1}^{n} M_{ij} \frac{\partial \theta_j'}{\partial \theta_k}$$

In matrix form, we have

$$\boldsymbol{I} = \boldsymbol{M} \boldsymbol{J}$$

where

$$J_{jk} = \frac{\partial \theta_j'}{\partial \theta_k}$$

$\rightarrow \qquad J = M^{-1}$

By eq(1.58), we have

$$\int d\theta_1 \ldots d\theta_n\, \theta_n \ldots \theta_1 = \int d\theta_1' \ldots d\theta_n'\, J \cdot \det M \cdot \theta_n' \ldots \theta_1'$$

Using

$$J = \det J = \det M^{-1}$$

we have

$$\int d\theta_1 \ldots d\theta_n\, \theta_n \ldots \theta_1 = \int d\theta_1' \ldots d\theta_n'\, \theta_n' \ldots \theta_1'$$

$$= \frac{\partial}{\partial \theta_1'} \ldots \frac{\partial}{\partial \theta_n'} \theta_n' \ldots \theta_1'$$

$$= 1$$

as required.

Mixed Change of Variables

Let θ, θ' & x, x' be anti-commuting & commuting variables, respectively.

A transformation $\{\theta, x\} \rightarrow \{\theta', x'\}$ that respect parities can be written as

$$x_a = x_a(x', \theta') \in \mathfrak{U}^+(\theta') \quad \& \qquad \theta_i = \theta_i(x', \theta') \in \mathfrak{U}^-(\theta') \qquad (1.60)$$

The differentials may be written as

$$d x_a = \frac{\partial x_a}{\partial x_b'} d x_b' + \frac{\partial x_a}{\partial \theta_i'} \delta \theta_i'$$

$$\delta \theta_i = \frac{\partial \theta_i}{\partial x_b'} d x_b' + \frac{\partial \theta_i}{\partial \theta_j'} \delta \theta_j'$$

where the differential of θ_i is denoted by $\delta \theta_i$ to avoid confusion with the $d\theta_i$ used in integrations that transforms like $\frac{\partial}{\partial \theta_i}$.

The tranformation can be written in matrix form by defining

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$A_{ab} = \frac{\partial x_a}{\partial x_b'} \qquad\qquad B_{ai} = \frac{\partial x_a}{\partial \theta_i'}$$

$$C_{ia} = \frac{\partial \theta_i}{\partial x_a'} \qquad\qquad D_{ij} = \frac{\partial \theta_i}{\partial \theta_j'}$$

so that

$$d\mathbf{x} = \mathbf{A}\, d\mathbf{x}' + \mathbf{B}\, \delta \boldsymbol{\theta}'$$

$$\delta \boldsymbol{\theta} = \mathbf{C}\, d\mathbf{x}' + \mathbf{D}\, \delta \boldsymbol{\theta}'$$

The jacobian J in

$$\int d x_1 \ldots d x_n \int d\theta_1 \ldots d\theta_n\, f(x, \theta) = \int d x_1' \ldots d x_n' \int d\theta_1' \ldots d\theta_n'\, J\, f(x', \theta')$$

does not equal to $\det M$ for two reasons:

1. B & $C \in \mathfrak{U}^-$ so that the meaning of $\det M$ itself is unclear.
2. $d\theta_i$ transforms opposite to $\delta \theta_i$.

J can be calculated in two steps.

Step 1:

$\{\theta, x\} \to \{\theta, x'\}$

$\to \quad \delta\boldsymbol{\theta} = 0 = \boldsymbol{C}\, d\,\boldsymbol{x}' + \boldsymbol{D}\,\delta\,\boldsymbol{\theta}'$

$\therefore \quad \delta\,\boldsymbol{\theta}' = -\frac{1}{\boldsymbol{D}}\boldsymbol{C}\, d\,\boldsymbol{x}'$

$$d\,\boldsymbol{x} = \left(\boldsymbol{A} - \boldsymbol{B}\frac{1}{\boldsymbol{D}}\boldsymbol{C}\right) d\,\boldsymbol{x}'$$

which gives a jacobian

$$J_1 = \det \left.\frac{\partial x_a}{\partial x_b'}\right|_\theta = \det\left(\boldsymbol{A} - \boldsymbol{B}\frac{1}{\boldsymbol{D}}\boldsymbol{C}\right) \in \mathfrak{U}^+ \qquad (1.61)$$

Step 2:

$\{\theta, x'\} \to \{\theta', x'\}$

Treating x' as a constant, eq(1.59) applies & we have

$$J_2 = \det\frac{\partial\theta_i'}{\partial\theta_j} = \det\left(\boldsymbol{D}^{-1}\right) = \frac{1}{\det\boldsymbol{D}} \in \mathfrak{U}^+ \qquad (1.62)$$

where we have used

$$\boldsymbol{D}_{ij} = \frac{\partial\theta_i}{\partial\theta_j'} \qquad \to \qquad \boldsymbol{D}_{ij}^{-1} = \frac{\partial\theta_i'}{\partial\theta_j}$$

because

$$\boldsymbol{D}_{ij}\boldsymbol{D}_{jk}^{-1} = \frac{\partial\theta_i}{\partial\theta_j'}\frac{\partial\theta_j'}{\partial\theta_k} = \frac{\partial\theta_i}{\partial\theta_k} = \delta_{ik}$$

Also,

$$\boldsymbol{D}\boldsymbol{D}^{-1} = \boldsymbol{I} \qquad \to \qquad (\det\boldsymbol{D})\det\left(\boldsymbol{D}^{-1}\right) = \det\boldsymbol{I} = 1$$

Combining the 2 steps, we have

$$J = J_1 J_2 = \det\left(\boldsymbol{A} - \boldsymbol{B}\frac{1}{\boldsymbol{D}}\boldsymbol{C}\right)\frac{1}{\det\boldsymbol{D}} \qquad (1.63)$$

J is also called the berezinian & denoted as

$J = \text{Ber}\,\boldsymbol{M}$

Trace of Mixed Matrices

For an infinitesimal transformation between commuting variables,

$x_a = x_a' + \varepsilon f_a(x')$

we have

$$\frac{\partial x_a}{\partial x_b'} = \delta_{ab} + \varepsilon\frac{\partial f_a}{\partial x_b'}$$

$$\to \qquad J = \det\frac{\partial x_a}{\partial x_b'}$$

$$= \det\left(\delta_{ab} + \varepsilon\frac{\partial f_a}{\partial x_b'}\right)$$

$$= \begin{vmatrix} 1 + \varepsilon\frac{\partial f_1}{\partial x_1'} & \varepsilon\frac{\partial f_1}{\partial x_2'} & \cdots \\ \varepsilon\frac{\partial f_2}{\partial x_1'} & 1 + \varepsilon\frac{\partial f_2}{\partial x_2'} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

Doing the Laplace expansion of the determinant, we see that anything involving an off-diagonal term is $O(\varepsilon^2)$. Terms of $O(\varepsilon)$ are therefore contained in the product of all diagonal elements. Expanding the product gives

$$J \approx 1 + \varepsilon \frac{\partial f_a}{\partial x_a{}'} + O(\varepsilon^2)$$
$$= 1 + \varepsilon \,\mathrm{tr} \frac{\partial f_a}{\partial x_b{}'} + O(\varepsilon^2)$$

For the mixed case, we have

$$x_a = x_a{}' + \varepsilon f_a(x', \theta') \qquad \& \qquad \theta_i = \theta_i{}' + \varepsilon\, \varphi_i(x', \theta') \tag{1.64}$$

Let

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$$
$$= \begin{pmatrix} \boldsymbol{I} + \varepsilon \boldsymbol{A}_1 + \ldots & \varepsilon \boldsymbol{B}_1 + \ldots \\ \varepsilon \boldsymbol{C}_1 + \ldots & \boldsymbol{I} + \varepsilon \boldsymbol{D}_1 + \ldots \end{pmatrix}$$
$$= I + \varepsilon \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{pmatrix} + O(\varepsilon^2)$$
$$= I + \varepsilon \boldsymbol{M}_1 + O(\varepsilon^2)$$

$$\to \qquad J = \det\left(\boldsymbol{A} - \boldsymbol{B} \frac{1}{\boldsymbol{D}} \boldsymbol{C} \right) \det\left(\boldsymbol{D}^{-1}\right)$$
$$= \det\left[\boldsymbol{I} + \varepsilon \boldsymbol{A}_1 + O(\varepsilon^2) \right] \det\left[\boldsymbol{I} - \varepsilon \boldsymbol{D}_1 + O(\varepsilon^2) \right]$$
$$= \left[1 + \varepsilon \,\mathrm{tr}\, \boldsymbol{A}_1 + O(\varepsilon^2) \right] \left[1 - \varepsilon \,\mathrm{tr}\, \boldsymbol{D}_1 + O(\varepsilon^2) \right]$$
$$= 1 + \varepsilon \left(\mathrm{tr}\, \boldsymbol{A}_1 - \mathrm{tr}\, \boldsymbol{D}_1 \right) + O(\varepsilon^2) \tag{1.65}$$

Using

$$\boldsymbol{A}_{a\,b} = \frac{\partial x_a}{\partial x_b{}'} \qquad \boldsymbol{D}_{i\,j} = \frac{\partial \theta_i}{\partial \theta_j{}'}$$

we have, to $O(\varepsilon^0)$,

$$(\boldsymbol{A}_1)_{a\,b} = \frac{\partial f_a}{\partial x_b{}'} = \frac{\partial f_a}{\partial x_b} \qquad (\boldsymbol{D}_1)_{i\,j} = \frac{\partial \varphi_i}{\partial \theta_j{}'} = \frac{\partial \varphi_i}{\partial \theta_j}$$

$$\to \qquad \mathrm{tr}\, \boldsymbol{A}_1 - \mathrm{tr}\, \boldsymbol{D}_1 = \frac{\partial f_a}{\partial x_a} - \frac{\partial \varphi_i}{\partial \theta_i}$$

Defing the supertrace as

$$\mathrm{Str}\, \boldsymbol{M}_1 = \mathrm{tr}\, \boldsymbol{A}_1 - \mathrm{tr}\, \boldsymbol{D}_1$$

eq(1.65) takes the form resembling the commuting case:

$$J = 1 + \varepsilon\, \mathrm{Str}\, \boldsymbol{M}_1 + O(\varepsilon^2) \tag{1.66}$$

$$\mathrm{Str}\, \boldsymbol{M}_1 \boldsymbol{M}_2 = \mathrm{Str}\left[\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{pmatrix} \right]$$
$$= \mathrm{Str} \begin{pmatrix} \boldsymbol{A}_1 \boldsymbol{A}_2 + \boldsymbol{B}_1 \boldsymbol{C}_2 & \ldots \\ \ldots & \boldsymbol{C}_1 \boldsymbol{B}_2 + \boldsymbol{D}_1 \boldsymbol{D}_2 \end{pmatrix}$$
$$= \mathrm{tr}(\boldsymbol{A}_1 \boldsymbol{A}_2 + \boldsymbol{B}_1 \boldsymbol{C}_2) - \mathrm{tr}(\boldsymbol{C}_1 \boldsymbol{B}_2 + \boldsymbol{D}_1 \boldsymbol{D}_2)$$

$$\mathrm{Str}\, \boldsymbol{M}_2 \boldsymbol{M}_1 = \mathrm{Str}\left[\begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{B}_2 \\ \boldsymbol{C}_2 & \boldsymbol{D}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1 \\ \boldsymbol{C}_1 & \boldsymbol{D}_1 \end{pmatrix} \right]$$
$$= \mathrm{Str} \begin{pmatrix} \boldsymbol{A}_2 \boldsymbol{A}_1 + \boldsymbol{B}_2 \boldsymbol{C}_1 & \ldots \\ \ldots & \boldsymbol{C}_2 \boldsymbol{B}_1 + \boldsymbol{D}_2 \boldsymbol{D}_1 \end{pmatrix}$$

$$= \mathrm{tr}(\boldsymbol{A}_2\, \boldsymbol{A}_1 + \boldsymbol{B}_2\, \boldsymbol{C}_1) - \mathrm{tr}(\boldsymbol{C}_2\, \boldsymbol{B}_1 + \boldsymbol{D}_2\, \boldsymbol{D}_1)$$

Since

$$\mathrm{tr}(\boldsymbol{A}\, \boldsymbol{B}) = \mathrm{tr}(\boldsymbol{B}\, \boldsymbol{A}) \qquad \& \qquad \mathrm{tr}(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{tr}\, \boldsymbol{A} + \mathrm{tr}\, \boldsymbol{B}$$

we have

$$\mathrm{Str}\, \boldsymbol{M}_2\, \boldsymbol{M}_1 = \mathrm{Str}\, \boldsymbol{M}_1\, \boldsymbol{M}_2$$