I.7. Gaussian Integrals with Grassmann Variables

Consider an algebra with $2\,n$ generators that can be written as

$$\{\theta_1,\ \ldots,\ \theta_n;\ \overline{\theta}_1,\ \ldots,\ \overline{\theta}_n\} \equiv \{\boldsymbol{\theta},\ \overline{\boldsymbol{\theta}}\}$$

where $\overline{\theta}_i$ is the result of applying the "conjugation" operation to θ_i, i.e.,

$$\overline{\theta}_i \equiv \overline{\theta_i}$$

such that

$$\overline{\overline{\theta}}_i = \overline{\overline{\theta_i}} = \theta_i \qquad \& \qquad \overline{\theta_i\,\theta_j} = \overline{\theta}_j\,\overline{\theta}_i$$

Gaussian Integral

Consider (implicit summation suspended)

$$\mathcal{Z}(\boldsymbol{M}) = \int d\,\theta_1\,d\,\overline{\theta}_1 \ldots d\,\theta_n\,d\,\overline{\theta}_n \exp\left(\sum_{i,j=1}^{n} \overline{\theta}_i\,M_{i\,j}\,\theta_j\right) \tag{1.67}$$

$$= \int d\,\boldsymbol{\theta}\,d\,\overline{\boldsymbol{\theta}} \exp\left(\overline{\boldsymbol{\theta}}\cdot\boldsymbol{M}\cdot\boldsymbol{\theta}\right)$$

where

$$d\,\boldsymbol{\theta}\,d\,\overline{\boldsymbol{\theta}} \equiv d\,\theta_1\,d\,\overline{\theta}_1 \ldots d\,\theta_n\,d\,\overline{\theta}_n$$

$$\equiv \prod_{i=1}^{n} d\,\theta_i{}'\,d\,\overline{\theta}_i$$

Using

$$\int d\,\theta_i\,d\,\overline{\theta}_i = 0$$

$$\int d\,\theta_i\,d\,\overline{\theta}_i\,\theta_i = \int d\,\theta_i\,d\,\overline{\theta}_i\,\overline{\theta}_i = 0$$

$$\int d\,\theta_i\,d\,\overline{\theta}_i\ \overline{\theta}_i\,\theta_i = 1$$

we see that $\mathcal{Z}(\boldsymbol{M})$ is just the coefficient of the term $\overline{\theta}_n\,\theta_n \ldots \overline{\theta}_1\,\theta_1$ in the expansion of $\exp\left(\sum_{i,j=1}^{n} \overline{\theta}_i\,M_{i\,j}\,\theta_j\right)$.

$$\exp\left(\sum_{i,j=1}^{n} \overline{\theta}_i\,M_{i\,j}\,\theta_j\right) = \prod_{i=1}^{n} \exp\left(\overline{\theta}_i \sum_{j=1}^{n} M_{i\,j}\,\theta_j\right)$$

$$= \prod_{i=1}^{n}\left(1 + \overline{\theta}_i \sum_{j=1}^{n} M_{i\,j}\,\theta_j\right) \qquad (\,\overline{\theta}_i^2 = 0\,)$$

Since each $\overline{\theta}_i\,\theta_j$ is accompanied by the factor $M_{i\,j}$, the coefficient of the $\overline{\theta}_n\,\theta_n \ldots \overline{\theta}_1\,\theta_1$ term must be

$$\sum_P (-)^P\,M_{n\,P(n)} \ldots M_{1\,P(1)} = \det \boldsymbol{M}$$

$$\rightarrow \qquad \mathcal{Z}(\boldsymbol{M}) = \det \boldsymbol{M} \tag{1.68}$$

which is to be contrasted with $\mathcal{Z}(\boldsymbol{M}) \propto \dfrac{1}{\sqrt{\det \boldsymbol{M}}}$ for commuting variables.

Another way to obtain eq(1.68) is set

$$\theta_i{}' = \sum_j M_{i\,j}\,\theta_j \tag{1.69}$$

so that

$$\mathcal{Z}(\boldsymbol{M}) = \det \boldsymbol{M} \int d\,\theta_1{}'\, d\,\overline{\theta}_1 \ldots d\,\theta_n{}'\, d\,\overline{\theta}_n \exp\left(\sum_{i=1}^{n} \overline{\theta}_i\, \theta_i{}' \right)$$

$$= \det \boldsymbol{M} \int d\,\theta_1{}'\, d\,\overline{\theta}_1 \ldots d\,\theta_n{}'\, d\,\overline{\theta}_n \prod_{i=1}^{n} \exp(\overline{\theta}_i\, \theta_i{}')$$

$$= \det \boldsymbol{M} \int d\,\theta_1{}'\, d\,\overline{\theta}_1 \ldots d\,\theta_n{}'\, d\,\overline{\theta}_n \prod_{i=1}^{n} (1 + \overline{\theta}_i\, \theta_i{}')$$

$$= \det \boldsymbol{M} \int \prod_{i=1}^{n} d\,\theta_i{}'\, d\,\overline{\theta}_i\, (1 + \overline{\theta}_i\, \theta_j{}')$$

$$= \det \boldsymbol{M}$$

General Gaussian Integrals

Let $\mathfrak{U}$ be spanned by the generators $\{\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}\}$ & $\mathfrak{U}' = \mathfrak{U}$ by $\{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}\}$.

Consider

$$\mathcal{Z}_G(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \int \prod_i d\,\theta_i\, d\,\overline{\theta}_i \exp\left[\sum_{i,j=1}^{n} M_{ij}\, \overline{\theta}_i\, \theta_j + \sum_{i=1}^{n} (\overline{\eta}_i\, \theta_i + \overline{\theta}_i\, \eta_i) \right] \qquad (1.70)$$

$$= \int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}} \exp(\overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})$$

where the integrand belong to $\mathfrak{U} \oplus \mathfrak{U}'$.

Setting

$$\theta_i = \theta_i{}' - \sum_j (M^{-1})_{ij}\, \eta_j \qquad \overline{\theta}_i = \overline{\theta}_i{}' - \sum_j \overline{\eta}_j\, (M^{-1})_{ji}$$

or

$$\boldsymbol{\theta} = \boldsymbol{\theta}' - \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta} \qquad \overline{\boldsymbol{\theta}} = \overline{\boldsymbol{\theta}}' - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1}$$

we have

$$\overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} = (\overline{\boldsymbol{\theta}}' - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1}) \cdot \boldsymbol{M} \cdot (\boldsymbol{\theta}' - \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$= \overline{\boldsymbol{\theta}}' \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}' - \overline{\boldsymbol{\theta}}' \cdot \boldsymbol{\eta} - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta}' + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}$$

$$\overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta} = \overline{\boldsymbol{\eta}} \cdot (\boldsymbol{\theta}' - \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) + (\overline{\boldsymbol{\theta}}' - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1}) \cdot \boldsymbol{\eta}$$

$$= \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta}' - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta} + \overline{\boldsymbol{\theta}}' \cdot \boldsymbol{\eta} - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}$$

$$\rightarrow \qquad \overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta} = \overline{\boldsymbol{\theta}}' \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}' - \overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}$$

Also

$$\frac{\partial}{\partial\,\theta_i} = \frac{\partial}{\partial\,\theta_i{}'} \qquad \frac{\partial}{\partial\,\overline{\theta}_i} = \frac{\partial}{\partial\,\overline{\theta}_i{}'}$$

$$\rightarrow \qquad \mathcal{Z}_G(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) \int d\,\boldsymbol{\theta}'\, d\,\overline{\boldsymbol{\theta}}' \exp(\overline{\boldsymbol{\theta}}' \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}')$$

Using eqs(1.67-8), we have

$$\mathcal{Z}_G(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) \det \boldsymbol{M}$$

$$= \exp\left(- \sum_{i,j=1}^{n} \overline{\eta}_i\, (M^{-1})_{ij}\, \eta_j \right) \det \boldsymbol{M} \qquad (1.71)$$

Since $\boldsymbol{\eta}$, $\overline{\boldsymbol{\eta}}$, $\boldsymbol{\theta}$ & $\overline{\boldsymbol{\theta}}$ have odd parity while $\overline{\theta}_i\, \eta_j$ & $\overline{\eta}_i\, \theta_j$ have even parity, we have

$$\exp(\overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta}) = 1 + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \frac{1}{2} (\overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta})^2 + \ldots$$

$$\rightarrow \quad \frac{\partial}{\partial \bar{\eta}_i} \exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta}) = \frac{\partial}{\partial \bar{\eta}_i} \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \frac{1}{2} \left[\left(\left(\frac{\partial}{\partial \bar{\eta}_i} \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} \right) \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} \right) + \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} \frac{\partial}{\partial \bar{\eta}_i} \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} \right] + \ldots$$

$$= \theta_i + \theta_i \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \ldots$$

$$= \theta_i \exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta})$$

$$= \exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta})\, \theta_i \tag{1.71a}$$

$$\exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}) = 1 + \bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta} + \frac{1}{2} (\bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})^2 + \ldots$$

$$\rightarrow \quad \frac{\partial}{\partial \eta_i} \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}) = -\bar{\boldsymbol{\theta}} \cdot \frac{\partial}{\partial \eta_i} \boldsymbol{\eta} + \frac{1}{2} \left[-\bar{\boldsymbol{\theta}} \cdot \left(\frac{\partial}{\partial \eta_i} \boldsymbol{\eta} \right) (\bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}) - (\bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}) \bar{\boldsymbol{\theta}} \cdot \left(\frac{\partial}{\partial \eta_i} \boldsymbol{\eta} \right) \right] + \ldots$$

$$= -\theta_i - \theta_i \bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta} + \ldots$$

$$= -\theta_i \exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta})$$

$$= -\exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta})\, \theta_i \tag{1.71b}$$

Hence,

$$\frac{\partial}{\partial \bar{\eta}_i} \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; \theta_i \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})$$

$$\rightarrow \quad \left. \frac{\partial}{\partial \bar{\eta}_i} \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) \right|_{\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} = 0} = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; \theta_i \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta})$$

$$= \det \boldsymbol{M}\, \langle \theta_i \rangle$$

where

$$\langle \theta_i \rangle = \frac{1}{\mathcal{Z}_G} \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; \theta_i \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta})$$

with

$$\mathcal{Z}_G = \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) \,|_{\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} = 0} = \det \boldsymbol{M}$$

Note that in a more concise notation

$$\left. \frac{\partial}{\partial \bar{\eta}_i} \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) \right|_{\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} = 0} \equiv \frac{\partial}{\partial \bar{\eta}_i} \mathcal{Z}_G$$

so that eq(1.72) can be written as

$$\frac{\partial}{\partial \bar{\eta}_i} \mathcal{Z}_G = \det \boldsymbol{M}\, \langle \theta_i \rangle \tag{1.72}$$

Similarly,

$$\frac{\partial}{\partial \eta_i} \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; (-\bar{\theta}_i) \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \bar{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \bar{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})$$

$$\rightarrow \quad \left. \frac{\partial}{\partial \eta_i} \mathcal{Z}_G(\boldsymbol{\eta}, \bar{\boldsymbol{\eta}}) \right|_{\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} = 0} = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; (-\bar{\theta}_i) \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta})$$

$$= \det \boldsymbol{M}\, \langle -\bar{\theta}_i \rangle \tag{1.73}$$

Wick's Theorem for Grassmann Integrals

In general, we define

$$\langle \bar{\theta}_{i_1} \theta_{j_1} \ldots \bar{\theta}_{i_k} \theta_{j_k} \rangle = \frac{1}{\mathcal{Z}_G} \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; \bar{\theta}_{i_1} \theta_{j_1} \ldots \bar{\theta}_{i_k} \theta_{j_k} \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta})$$

$$= \frac{1}{\det \boldsymbol{M}} \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; \bar{\theta}_{i_1} \theta_{j_1} \ldots \bar{\theta}_{i_k} \theta_{j_k} \exp(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}) \tag{1.74}$$

$$= \frac{1}{\det \boldsymbol{M}}\left[\left(-\frac{\partial}{\partial \eta_{i_1}}\right)\frac{\partial}{\partial \overline{\eta}_{j_1}} \cdots \left(-\frac{\partial}{\partial \eta_{i_k}}\right)\frac{\partial}{\partial \overline{\eta}_{j_k}} \mathcal{Z}_G(\eta, \overline{\eta})\right]_{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}=0}$$

$$= \frac{1}{\det \boldsymbol{M}}\left[\frac{\partial}{\partial \overline{\eta}_{j_1}}\frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \overline{\eta}_{j_k}}\frac{\partial}{\partial \eta_{i_k}} \mathcal{Z}_G(\eta, \overline{\eta})\right]_{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}=0} \tag{1.75}$$

$$= \left[\frac{\partial}{\partial \overline{\eta}_{j_1}}\frac{\partial}{\partial \eta_{i_1}} \cdots \frac{\partial}{\partial \overline{\eta}_{j_k}}\frac{\partial}{\partial \eta_{i_k}} \exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})\right]_{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}=0} \tag{1.76}$$

$$\frac{\partial}{\partial \eta_b}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) = \frac{\partial}{\partial \eta_b}\exp\left(-\sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{ij}\, \eta_j\right)$$

$$= -\left(\frac{\partial}{\partial \eta_b}\sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{ij}\, \eta_j\right)\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$= \left(\sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{ij} \frac{\partial}{\partial \eta_b}\eta_j\right)\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$= \sum_{i=1}^{n} \overline{\eta}_i (M^{-1})_{ib} \exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$\frac{\partial}{\partial \overline{\eta}_a}\frac{\partial}{\partial \eta_b}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) = \frac{\partial}{\partial \overline{\eta}_a}\left[\sum_{i=1}^{n} \overline{\eta}_i (M^{-1})_{ib} \exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})\right]$$

$$= \left((M^{-1})_{ab} - \sum_{i=1}^{n} \overline{\eta}_i (M^{-1})_{ib} \frac{\partial}{\partial \overline{\eta}_a}\right)\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$= \left((M^{-1})_{ab} + \sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{ib} (M^{-1})_{aj}\, \eta_j\right)\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$\rightarrow \qquad \left\langle \overline{\theta}_a\, \theta_b \right\rangle = \frac{\partial}{\partial \overline{\eta}_b}\frac{\partial}{\partial \eta_a}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})\Big|_{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}=0} = (M^{-1})_{ba}$$

$$\frac{\partial}{\partial \overline{\eta}_c}\frac{\partial}{\partial \eta_d}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) = \left((M^{-1})_{cd} + \sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{id} (M^{-1})_{cj}\, \eta_j\right)\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$\rightarrow \qquad \frac{\partial}{\partial \overline{\eta}_a}\frac{\partial}{\partial \eta_b}\frac{\partial}{\partial \overline{\eta}_c}\frac{\partial}{\partial \eta_d}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta}) = \Big[-(M^{-1})_{ad} (M^{-1})_{cb} +$$

$$\left((M^{-1})_{cd} + \sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{id} (M^{-1})_{cj}\, \eta_j\right)\frac{\partial}{\partial \overline{\eta}_a}\frac{\partial}{\partial \eta_b}\Big]\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$= \Big\{-(M^{-1})_{ad} (M^{-1})_{cb} +$$

$$\left[(M^{-1})_{cd} + \sum_{i,j=1}^{n} \overline{\eta}_i (M^{-1})_{id} (M^{-1})_{cj}\, \eta_j\right]$$

$$\times\left[(M^{-1})_{ab} + \sum_{k,l=1}^{n} \overline{\eta}_k (M^{-1})_{kb} (M^{-1})_{al}\, \eta_l\right]\Big\}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})$$

$$\therefore \qquad \left\langle \overline{\theta}_b\, \theta_a\, \overline{\theta}_d\, \theta_c \right\rangle = \frac{\partial}{\partial \overline{\eta}_a}\frac{\partial}{\partial \eta_b}\frac{\partial}{\partial \overline{\eta}_c}\frac{\partial}{\partial \eta_d}\exp(-\overline{\boldsymbol{\eta}} \cdot \boldsymbol{M}^{-1} \cdot \boldsymbol{\eta})\Big|_{\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}=0}$$

$$= (M^{-1})_{ab}(M^{-1})_{cd} - (M^{-1})_{ad}(M^{-1})_{cb}$$

$$= \det \begin{vmatrix} (M^{-1})_{ab} & (M^{-1})_{ad} \\ (M^{-1})_{cb} & (M^{-1})_{cd} \end{vmatrix}$$

$$= \det \begin{vmatrix} \langle \overline{\theta}_b \theta_a \rangle & \langle \overline{\theta}_d \theta_a \rangle \\ \langle \overline{\theta}_b \theta_c \rangle & \langle \overline{\theta}_d \theta_c \rangle \end{vmatrix}$$

$$\rightarrow \qquad \langle \overline{\theta}_{i_1} \theta_{j_1} \overline{\theta}_{i_2} \theta_{j_2} \rangle = (M^{-1})_{j_1 i_1}(M^{-1})_{j_2 i_2} - (M^{-1})_{j_1 i_2}(M^{-1})_{j_2 i_1}$$

$$= \det \begin{vmatrix} (M^{-1})_{j_1 i_1} & (M^{-1})_{j_2 i_1} \\ (M^{-1})_{j_1 i_2} & (M^{-1})_{j_2 i_2} \end{vmatrix} = \det A_{ab} \qquad \text{where} \quad A_{ab} = M^{-1}_{j_b i_a}$$

$$= \det \begin{vmatrix} \langle \overline{\theta}_{i_1} \theta_{j_1} \rangle & \langle \overline{\theta}_{i_1} \theta_{j_2} \rangle \\ \langle \overline{\theta}_{i_2} \theta_{j_1} \rangle & \langle \overline{\theta}_{i_2} \theta_{j_2} \rangle \end{vmatrix} = \det B_{ab} \qquad \text{where} \quad B_{ab} = \langle \overline{\theta}_{i_a} \theta_{j_b} \rangle$$

Note that $\det A = \det A^T$ so that we can define $A_{ab} = M^{-1}_{j_a i_b}$ & $B_{ab} = \langle \overline{\theta}_{i_b} \theta_{j_a} \rangle$ instead.

The actual derivation is rather tedious so we'll just take a leap of faith & write

$$\langle \overline{\theta}_{i_1} \theta_{j_1} \ldots \overline{\theta}_{i_k} \theta_{j_k} \rangle = \det \begin{vmatrix} (M^{-1})_{j_1 i_1} & \cdots & (M^{-1})_{j_k i_1} \\ \vdots & \ddots & \vdots \\ (M^{-1})_{j_1 i_k} & \cdots & (M^{-1})_{j_k i_k} \end{vmatrix}$$

$$= \det \begin{vmatrix} \langle \overline{\theta}_{i_1} \theta_{j_1} \rangle & \cdots & \langle \overline{\theta}_{i_1} \theta_{j_k} \rangle \\ \vdots & \ddots & \vdots \\ \langle \overline{\theta}_{i_k} \theta_{j_1} \rangle & \cdots & \langle \overline{\theta}_{i_k} \theta_{j_k} \rangle \end{vmatrix}$$

$$= \sum_P (-)^P (M^{-1})_{j_{P(1)} i_1} \ldots (M^{-1})_{j_{P(k)} i_k} \tag{1.77}$$

Perturbative Expansion

Let

$$\mathcal{Z}(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \int \prod_i d\theta_i \, d\overline{\theta}_i \exp\Big[\sum_{i,j=1}^{n} M_{ij} \overline{\theta}_i \theta_j + V(\overline{\boldsymbol{\theta}}, \boldsymbol{\theta}) + \sum_{i=1}^{n} (\overline{\eta}_i \theta_i + \overline{\theta}_i \eta_i) \Big] \tag{1.78}$$

$$= \int d\boldsymbol{\theta} \, d\overline{\boldsymbol{\theta}} \exp[\, \overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + V(\overline{\boldsymbol{\theta}}, \boldsymbol{\theta}) + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}]$$

Since $\overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}$, $\overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta}$ & $\overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta}$ are all commuting variables, we have

$$\mathcal{Z}(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \int d\boldsymbol{\theta} \, d\overline{\boldsymbol{\theta}} \exp[V(\overline{\boldsymbol{\theta}}, \boldsymbol{\theta})] \exp(\, \overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})$$

Expand $\exp[V(\overline{\boldsymbol{\theta}}, \boldsymbol{\theta})]$ in a power series & then apply eqs(1.71a,b) to get

$$\mathcal{Z}(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = \exp\Big[V\Big(-\frac{\partial}{\partial \boldsymbol{\eta}}, \frac{\partial}{\partial \overline{\boldsymbol{\eta}}}\Big)\Big] \int d\boldsymbol{\theta} \, d\overline{\boldsymbol{\theta}} \exp(\, \overline{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\eta}} \cdot \boldsymbol{\theta} + \overline{\boldsymbol{\theta}} \cdot \boldsymbol{\eta})$$

$$= \exp\Big[V\Big(-\frac{\partial}{\partial \boldsymbol{\eta}}, \frac{\partial}{\partial \overline{\boldsymbol{\eta}}}\Big)\Big] \mathcal{Z}_G(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) \tag{1.79}$$

The Pfaffian

More generally, if the $2n$ generators are not related by conjugation, we have

$$\mathcal{Z}(\boldsymbol{A}) = \int d\,\theta_{2n}\, d\,\theta_{2n-1} \ldots d\,\theta_2\, d\,\theta_1 \exp\left(\frac{1}{2}\sum_{i,j=1}^{2n} \theta_i A_{ij}\, \theta_j\right) \tag{1.80}$$

Since

$$\sum_{i,j} \theta_i A_{ij}\, \theta_j = \sum_{i,j} \theta_j A_{ji}\, \theta_i$$
$$= -\sum_{i,j} \theta_i A_{ji}\, \theta_j$$

the symmetric part of $\boldsymbol{A}$ does not contribute to the sum. We can therefore assume $\boldsymbol{A}$ to be antisymmetric, i.e.,

$$A_{ij} = -A_{ji} \tag{1.81}$$

Note that

$$A_{ii} = -A_{ii} \qquad \rightarrow \qquad A_{ii} = 0$$

which means every diagonal element of an antisymmetric $\boldsymbol{A}$ vanishes.

As before, expanding the exponential in a power series shows that only the n^{th} power term is able to give a non-vanishing integral, i.e.,

$$\mathcal{Z}(\boldsymbol{A}) = \frac{1}{2^n\, n!} \int d\,\theta_{2n} \ldots d\,\theta_1 \left(\sum_{i,j=1}^{2n} \theta_i A_{ij}\, \theta_j\right)^n \tag{1.82}$$

Actually, only monomial terms in $\left(\sum_{i,j=1}^{2n} \theta_i A_{ij}\, \theta_j\right)^n$ that are proportional to $\theta_1 \ldots \theta_{2n}$ contribute. Since $\theta_i\, \theta_j$ comes with A_{ij}, we have

$$\mathcal{Z}(\boldsymbol{A}) = \frac{1}{2^n\, n!} \sum_P (-)^P A_{P(1)\,P(2)} \ldots A_{P(2n-1)\,P(2n)} \tag{1.83}$$

where P denotes any one of the $(2\,n)!$ permutations with parity $(-)^P$ so that

$$P\{1, 2, \ldots, 2\,n\} = \{P(1), \ldots, P(2\,n)\}$$

Note that while $\boldsymbol{A}$ is an $(2\,n)\times(2\,n)$ matrix, eq(1.83) is a homogeneous polynomial of degree n in A_{ij}.

In matrix algebra, the pfaffian of an antisymmetric matrix $\boldsymbol{A}$ is defined as a polynomial whose square is equal to $\det \boldsymbol{A}$, i.e.,

$$\text{Pf}(\boldsymbol{A})^2 = \det(\boldsymbol{A}) \tag{1.85}$$

It is interesting that

$$\text{Pf}(\boldsymbol{A}) = \mathcal{Z}(\boldsymbol{A}) \tag{1.84}$$

For example, with $n = 1$, we have

$$\boldsymbol{A} = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}$$

Using eq(1.84), we have

$$\text{Pf}(\boldsymbol{A}) = \frac{1}{2}\,(A_{12} - A_{21}) = A_{12}$$

$$\rightarrow \qquad \text{Pf}(\boldsymbol{A})^2 = (A_{12})^2 = \det \boldsymbol{A}$$

The proof of eq(1.84) is as follows. By eq(1.80), we have

$$\mathcal{Z}(\boldsymbol{A})^2 = \int d\,\theta_{2n} \ldots d\,\theta_1\, d\,\theta_{2n}' \ldots d\,\theta_1' \exp\left(\frac{1}{2}\sum_{i,j=1}^{2n} \left(\theta_i A_{ij}\, \theta_j + \theta_i' A_{ij}\, \theta_j'\right)\right) \tag{1.86}$$

Let

$$\eta_k = \frac{1}{\sqrt{2}}(\theta_k + i\,\theta_k') \qquad \bar{\eta}_k = \frac{1}{\sqrt{2}}(\theta_k - i\,\theta_k')$$

$$\to \qquad \theta_k = \frac{1}{\sqrt{2}}(\eta_k + \bar{\eta}_k) \qquad \theta_k' = \frac{1}{i\sqrt{2}}(\eta_k - \bar{\eta}_k)$$

$$\therefore \qquad \theta_i A_{ij}\theta_j = \frac{1}{2}(\eta_i + \bar{\eta}_i) A_{ij}(\eta_j + \bar{\eta}_j) = \frac{1}{2}(\eta_i A_{ij}\eta_j + \eta_i A_{ij}\bar{\eta}_j + \bar{\eta}_i A_{ij}\eta_j + \bar{\eta}_i A_{ij}\bar{\eta}_j)$$

$$\theta_i' A_{ij}\theta_j' = -\frac{1}{2}(\eta_i - \bar{\eta}_i) A_{ij}(\eta_j - \bar{\eta}_j) = \frac{1}{2}(-\eta_i A_{ij}\eta_j + \eta_i A_{ij}\bar{\eta}_j + \bar{\eta}_i A_{ij}\eta_j - \bar{\eta}_i A_{ij}\bar{\eta}_j)$$

$$\to \qquad \theta_i A_{ij}\theta_j + \theta_i' A_{ij}\theta_j' = \eta_i A_{ij}\bar{\eta}_j + \bar{\eta}_i A_{ij}\eta_j$$
$$= \bar{\eta}_j A_{ji}\eta_i + \bar{\eta}_i A_{ij}\eta_j$$

$$\therefore \qquad \frac{1}{2}\sum_{i,j=1}^{2n}(\theta_i A_{ij}\theta_j + \theta_i' A_{ij}\theta_j') = \frac{1}{2}\sum_{i,j=1}^{2n}(\bar{\eta}_j A_{ji}\eta_i + \bar{\eta}_i A_{ij}\eta_j)$$
$$= \sum_{i,j=1}^{2n}\bar{\eta}_i A_{ij}\eta_j$$

$$\frac{\partial}{\partial\theta_k} = \frac{\partial\eta_k}{\partial\theta_k}\frac{\partial}{\partial\eta_k} + \frac{\partial\bar{\eta}_k}{\partial\theta_k}\frac{\partial}{\partial\bar{\eta}_k} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\eta_k} + \frac{\partial}{\partial\bar{\eta}_k}\right)$$

$$\frac{\partial}{\partial\theta_k'} = \frac{\partial\eta_k}{\partial\theta_k'}\frac{\partial}{\partial\eta_k} + \frac{\partial\bar{\eta}_k}{\partial\theta_k'}\frac{\partial}{\partial\bar{\eta}_k} = i\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\eta_k} - \frac{\partial}{\partial\bar{\eta}_k}\right)$$

$$\to \qquad \frac{\partial}{\partial\theta_k}\frac{\partial}{\partial\theta_k'} = \frac{1}{2}i\left(-\frac{\partial}{\partial\eta_k}\frac{\partial}{\partial\bar{\eta}_k} + \frac{\partial}{\partial\bar{\eta}_k}\frac{\partial}{\partial\eta_k}\right) = -i\frac{\partial}{\partial\eta_k}\frac{\partial}{\partial\bar{\eta}_k}$$

Consider

$$I = \int d\theta_{2n}\, d\theta_{2n-1} \ldots d\theta_1\, d\theta_{2n}'\, d\theta_{2n-1}' \ldots d\theta_1'\, f$$
$$= (-)^{2n-1}\int d\theta_{2n}\, d\theta_{2n}'\, d\theta_{2n-1} \ldots d\theta_1\, d\theta_{2n-1}' \ldots d\theta_1' f$$
$$= (-)^{2n-1}(-)^{2n-2}\int d\theta_{2n}\, d\theta_{2n}'\, d\theta_{2n-1}\, d\theta_{2n-1}' \ldots d\theta_1\, d\theta_{2n-2}' \ldots d\theta_1' f$$
$$\vdots$$
$$= (-)^s\int d\theta_{2n}\, d\theta_{2n}' \ldots d\theta_1\, d\theta_1' f$$

with

$$s = \sum_{k=1}^{2n-1} = \frac{1}{2}(2n-1)(2n) = n(2n-1)$$

$$\to \qquad (-)^s = (-)^n$$

$$\therefore \qquad I = (-)^n\frac{\partial}{\partial\theta_{2n}}\frac{\partial}{\partial\theta_{2n}'}\ldots\frac{\partial}{\partial\theta_1}\frac{\partial}{\partial\theta_1'}f$$
$$= (-)^n(-i)^{2n}\frac{\partial}{\partial\eta_{2n}}\frac{\partial}{\partial\bar{\eta}_{2n}}\ldots\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\bar{\eta}_1}f$$
$$= \frac{\partial}{\partial\eta_{2n}}\frac{\partial}{\partial\bar{\eta}_{2n}}\ldots\frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\bar{\eta}_1}f$$
$$= \int d\eta_{2n}\, d\bar{\eta}_{2n} \ldots d\eta_1\, d\bar{\eta}_1\, f$$

$$= \int d\eta_1 \, d\bar{\eta}_1 \ldots d\eta_{2n} \, d\bar{\eta}_{2n} \, f$$

where the last equality is obtained using the fact $d\eta_k \, d\bar{\eta}_k$ is a commuting variable.

Hence,

$$\mathcal{Z}(\boldsymbol{A})^2 = \int d\eta_1 \, d\bar{\eta}_1 \ldots d\eta_{2n} \, d\bar{\eta}_{2n} \exp\left(\sum_{i,j=1}^{2n} \bar{\eta}_i A_{ij} \eta_j\right)$$

$$= \int d\boldsymbol{\eta} \, d\bar{\boldsymbol{\eta}} \exp(\bar{\boldsymbol{\eta}} \cdot \boldsymbol{A} \cdot \boldsymbol{\eta})$$

$$= \det \boldsymbol{A} \qquad \text{QED}$$

Wick's Theorem

Another way to write eq(1.77) is

$$\langle \bar{\theta}_{i_1} \theta_{j_1} \ldots \bar{\theta}_{i_k} \theta_{j_k} \rangle = \sum_P (-)^P \langle \bar{\theta}_{i_1} \theta_{j_{P(1)}} \rangle \ldots \langle \bar{\theta}_{i_k} \theta_{j_{P(k)}} \rangle \tag{1.89}$$

which is just the Wick's theorem for Gaussian of anti-commuting variables.