1.8. Legendre Transformation

Consider a real function $W(\boldsymbol{h})$ of n variables h_i such that the matrix with elements $\partial_i \partial_j W$ exists & is positive, i.e.,

$$\sum_{i,j} v_i \left(\frac{\partial^2 W}{\partial h_i \partial h_j} \right) v_j > 0 \qquad \forall\, \boldsymbol{v} \text{ with } |\boldsymbol{v}| > 0$$

The Legendre transform of $W(\boldsymbol{h})$ is a function $\Gamma(\boldsymbol{m})$ defined by

$$W(\boldsymbol{h}) + \Gamma(\boldsymbol{m}) = \boldsymbol{h} \cdot \boldsymbol{m} = \sum_i h_i m_i \tag{1.90a}$$

where

$$m_i = \frac{\partial W}{\partial h_i} \tag{1.90b}$$

Note that the positivity of $\partial_i \partial_j W$ means that eq(1.90b) is invertible to give $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{m})$.

The Legendre Transformation is Involutive

$\dfrac{\partial}{\partial m_i}$ eq(1.90a) gives

$$\frac{\partial W}{\partial m_i} + \frac{\partial \Gamma}{\partial m_i} = h_i + \sum_j \frac{\partial h_j}{\partial m_i} m_j$$

Using

$$\frac{\partial W}{\partial m_i} = \sum_j \frac{\partial h_j}{\partial m_i} \frac{\partial W}{\partial h_j}$$

we have

$$\frac{\partial \Gamma}{\partial m_i} - h_i + \sum_j \frac{\partial h_j}{\partial m_i} \left(\frac{\partial W}{\partial h_j} - m_j \right) = 0 \tag{1.90c}$$

Now

$$\begin{aligned} \frac{\partial W}{\partial h_j} - m_j &= \frac{\partial}{\partial h_j} \left(W - \sum_i m_i h_i \right) \Bigg|_{\boldsymbol{m}\,\text{fixed}} \\ &= \frac{\partial \Gamma(\boldsymbol{m})}{\partial h_j} \Bigg|_{\boldsymbol{m}\,\text{fixed}} \qquad \text{[See eq(1.90a)]} \\ &= 0 \end{aligned}$$

Eq(1.90c) thus becomes

$$h_i = \frac{\partial \Gamma}{\partial m_i} \tag{1.91}$$

$$\rightarrow \qquad \frac{\partial^2 \Gamma}{\partial m_j \partial m_i} = \frac{\partial h_i}{\partial m_j}$$

Eq(1.90b) gives

$$\frac{\partial^2 W}{\partial h_j \partial h_i} = \frac{\partial m_i}{\partial h_j}$$

Hence,

$$\sum_k \frac{\partial^2 W}{\partial h_i \partial h_k} \frac{\partial^2 \Gamma}{\partial m_k \partial m_j} = \sum_k \frac{\partial m_k}{\partial h_i} \frac{\partial h_j}{\partial m_k}$$

$$= \frac{\partial h_j}{\partial h_i} = \delta_{ij}$$

which means the matrix $\partial_i \partial_j \Gamma(\boldsymbol{m})$ is the inverse of $\partial_i \partial_j W(\boldsymbol{h})$.

Hence, $\partial_i \partial_j \Gamma(\boldsymbol{m})$ is also positive.

Proof:

A real symmetric matrix $\boldsymbol{A}$ is positive means that its eigenvalues are all real & positive. (Proof of this is straightforward & thus skipped here.)

Let $\boldsymbol{u}_i$ be the eigenvector of $\boldsymbol{A}$ with eigenvalue $\lambda_i > 0$, i.e.,

$$\boldsymbol{A}\, \boldsymbol{u}_i = \lambda_i\, \boldsymbol{u}_i$$

$\boldsymbol{A}^{-1}$ exists since $\lambda_i \neq 0\ \forall\ i$.

$\rightarrow \qquad \boldsymbol{u}_i = \lambda_i\ \boldsymbol{A}^{-1}\, \boldsymbol{u}_i$

$\therefore \qquad \boldsymbol{A}^{-1}\, \boldsymbol{u}_i = \frac{1}{\lambda_i}\, \boldsymbol{u}_i$

i.e., $\boldsymbol{u}_i$ is also the eigenvector of $\boldsymbol{A}^{-1}$ with eigenvalue $\frac{1}{\lambda_i} > 0$. This holds for all i so that $\boldsymbol{A}^{-1}$ is positive.

Stationarity of $W + \Gamma$

Let $W(\boldsymbol{h})$ & hence $\Gamma(\boldsymbol{m})$ depend on a parameter μ, i.e.,

$$W = W\,[\,\boldsymbol{h}(\mu),\ \mu\,] \qquad \& \qquad \Gamma = \Gamma[\,\boldsymbol{m}(\mu),\ \mu\,]$$

$\frac{d}{d\mu}$ eq(1.90a) then gives

$$\frac{\partial W}{\partial \mu} + \sum_i \frac{d h_i}{d\mu} \frac{\partial W}{\partial h_i} + \frac{\partial \Gamma}{\partial \mu} + \sum_i \frac{d m_i}{d\mu} \frac{\partial \Gamma}{\partial m_i} = \sum_i \left(\frac{d h_i}{d\mu} m_i + h_i \frac{d m_i}{d\mu} \right)$$

Since (see eq(1.90b) & eq(1.91))

$$m_i = \frac{\partial W}{\partial h_i} \qquad \& \qquad h_i = \frac{\partial \Gamma}{\partial m_i}$$

we have

$$\frac{\partial W}{\partial \mu} + \frac{\partial \Gamma}{\partial \mu} = 0$$

or, following Zinn-Justin's more involved notations,

$$\left.\frac{\partial W}{\partial \mu}\right|_{\boldsymbol{h}} + \left.\frac{\partial \Gamma}{\partial \mu}\right|_{\boldsymbol{m}} = 0 \qquad (1.93)$$

Legendre Transformation and Real Steepest Descent Method

In contrast to the complex case, the method of real steepest descent approximates the integral

$$I = \int d^n x\, e^{-f(\boldsymbol{x})}$$

by replacing f by a power series expansion about its *minimum*, i.e.,

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \frac{1}{2} \sum_{i,j} \partial_i \partial_j\, f(\boldsymbol{x}_0)\, x_i\, x_j$$

where

$$\partial_i f(\boldsymbol{x}_0) = 0$$

& the matrix $\partial_i \partial_j f(\boldsymbol{x}_0)$ is positive. Thus,

$$I \approx e^{-f(\boldsymbol{x}_0)} g$$

where g is the Gaussian integral

$$g = \int d^n x \exp\left[-\frac{1}{2} \sum_{i,j} \partial_i \partial_j f(\boldsymbol{x}_0) x_i x_j\right]$$

For a distribution e^{-A}, the generating function $W(\boldsymbol{h})$ of the cumulants is given by

$$e^{W(\boldsymbol{b})} = \int d^n x \, e^{-A(\boldsymbol{x}) + \boldsymbol{b} \cdot \boldsymbol{x}} \qquad (1.94)$$

In a steepest decent approximation, we set

$$-\partial_i A(\boldsymbol{x}_0) + b_i = 0 \qquad (1.95)$$

so that

$$e^{W(\boldsymbol{b})} = e^{-A(\boldsymbol{x}_0) + \boldsymbol{b} \cdot \boldsymbol{x}_0 + c}$$

where

$$c = \ln \int d^n x \exp\left[-\frac{1}{2} \sum_{i,j} \partial_i \partial_j A(\boldsymbol{x}_0) x_i x_j\right]$$

→ $\quad W(\boldsymbol{b}) = -A(\boldsymbol{x}_0) + \boldsymbol{b} \cdot \boldsymbol{x}_0 + c$

Usually, the constant does not affect the behavior of the system & can be dropped of absorbed in W. Dropping the subscript 0, we have

$$W(\boldsymbol{b}) + A(\boldsymbol{x}) = \boldsymbol{b} \cdot \boldsymbol{x} \qquad (1.96)$$

which can be interpreted as a Legendre transform between W & A.

Adaptation to the case of complex or Grassmann variables should be easily done.