I.9. Generating Functionals. Functional Derivatives. Determinants

Generating Functional. Functional Differentiation.

Generating functional is useful when dealing with correlation functions.

Let

$$\left\{ F^{(n)}(x_1, \ \ldots, x_n) \ \middle| \ n = 0, 1, \ \ldots \right\}$$

be a set of functionals symmetric in their arguments.

The generating functional $\mathcal{F}(f)$ of $F^{(n)}$ is defined as

$$\mathcal{F}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\,x_1 \ldots d\,x_n\, F^{(n)}(x_1, \ \ldots, x_n)\, f(x_1) \ldots f(x_n) \tag{1.97}$$

$$= 1 + \int d\,x\, F^{(1)}(x)\, f(x) + \frac{1}{2} \int d\,x_1\, d\,x_2\, F^{(2)}(x_1, x_2)\, f(x_1)\, f(x_2) + \ldots$$

where f is any function of one variable.

The functional derivative $\dfrac{\delta}{\delta f(x)}$ is a differentiation satisfying

1. Linearity:

$$\frac{\delta}{\delta f(x)}[\mathcal{F}_1(f) + \mathcal{F}_2(f)] = \frac{\delta}{\delta f(x)} \mathcal{F}_1(f) + \frac{\delta}{\delta f(x)} \mathcal{F}_2(f) \tag{1.98a}$$

2. Leibnitz rule:

$$\frac{\delta}{\delta f(x)}[\mathcal{F}_1(f)\, \mathcal{F}_2(f)] = \left[\frac{\delta}{\delta f(x)} \mathcal{F}_1(f) \right] \mathcal{F}_2(f) + \mathcal{F}_1(f) \frac{\delta}{\delta f(x)} \mathcal{F}_2(f) \tag{1.98a}$$

Furthermore,

$$\frac{\delta}{\delta f(y)} f(x) = \delta(x - y) \tag{1.99}$$

so that, for example,

$$\frac{\delta}{\delta f(y)} \mathcal{F}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\,x_1 \ldots d\,x_n\, F^{(n)}(x_1, \ \ldots, x_n) \frac{\delta}{\delta f(y)} [\, f(x_1) \ldots f(x_n)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \int \prod_{i \neq j}^{n} d\,x_i\, F^{(n)}(x_1, \ \ldots, x_j \to y, \ \ldots, x_n) \prod_{k \neq j} f(x_k) \Big]$$

Renaming the indices $\{j+1, \ \ldots, n\}$ as $\{j, \ \ldots, n-1\}$, we have

$$\left\{ x_1, \ \ldots, x_j \to y, \ \ldots, x_n \right\} = \{ y, x_1, \ \ldots, x_{n-1} \}$$

$$\prod_{i \neq j}^{n} d\,x_i = \prod_{i=1}^{n-1} d\,x_i \qquad \& \qquad \prod_{k \neq j} f(x_k) = \prod_{k=1}^{n-1} f(x_k)$$

so that

$$\frac{\delta}{\delta f(y)} \mathcal{F}(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=1}^{n} \int \prod_{i=1}^{n-1} d\,x_i\, F^{(n-1)}(y, x_1, \ \ldots, x_{n-1}) \prod_{k=1}^{n-1} f(x_k) \Big]$$

Since the integrals are independent of j, we have

$$\frac{\delta}{\delta f(y)} \mathcal{F}(f) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n-1} d\,x_i\, F^{(n-1)}(y, x_1, \ \ldots, x_{n-1}) \prod_{k=1}^{n-1} f(x_k)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d x_i F^{(n+1)}(y, x_1, \ldots, x_n) \prod_{k=1}^{n} f(x_k) \tag{1.100}$$

$$= F^{(1)}(y) + \int d x F^{(2)}(y, x) f(x) + \ldots$$

Hence,

$$F^{(1)}(y) = \frac{\delta}{\delta f(y)} \mathcal{F}(f) \bigg|_{f=0}$$

$\frac{\delta}{\delta f}$ eq(1.100) again gives

$$\frac{\delta}{\delta f(y_1)} \frac{\delta}{\delta f(y_2)} \mathcal{F}(f) = F^{(2)}(y_1, y_2) + \ldots.$$

$$\rightarrow \qquad F^{(2)}(y_1, y_2) = \frac{\delta}{\delta f(y_1)} \frac{\delta}{\delta f(y_2)} \mathcal{F}(f) \bigg|_{f=0}$$

Continuing this way gives

$$F^{(p)}(y_1, \ldots, y_p) = \frac{\delta}{\delta f(y_1)} \ldots \frac{\delta}{\delta f(y_p)} \mathcal{F}(f) \bigg|_{f=0}$$

Determinants of Operators

Gaussian integrals give determinants. For example,

$$\int d^n x \exp\left(-\frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}}$$

$$\int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}} \exp\left(\bar{\boldsymbol{\theta}} \cdot \boldsymbol{M} \cdot \boldsymbol{\theta}\right) = \det \boldsymbol{M} \qquad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$$

As $n \to \infty$, the matrices $\boldsymbol{A}$ & $\boldsymbol{M}$ become operators.
The associated Gaussian integrals thus give determinants of operators.

For any matrix $\boldsymbol{M}$, we have

$$\det \boldsymbol{M} = \prod_i \lambda_i$$

If $\boldsymbol{M}$ is positive, we have

$$\ln \det \boldsymbol{M} = \sum_i \ln \lambda_i$$

Since the eigenvalues of $\ln \boldsymbol{M}$ are simply $\ln \lambda_i$, we have

$$\ln \det \boldsymbol{M} = \operatorname{tr} \ln \boldsymbol{M} \tag{1.101}$$

which should be valid for $n \to \infty$, and hence any positive operators, provided $\prod_i \lambda_i$ remains finite as $n \to \infty$.

If we can write

$$\boldsymbol{M} = \boldsymbol{I} + \boldsymbol{K} \quad \rightarrow \quad M(x, y) = \delta(x - y) + K(x, y)$$

then

$$\ln \boldsymbol{M} = \ln(\boldsymbol{I} + \boldsymbol{K})$$

$$= \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m} \boldsymbol{K}^m$$

$$\rightarrow \qquad \ln\det \boldsymbol{M} = \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m} \operatorname{tr} \boldsymbol{K}^m$$

$$\rightarrow \sum_{m=1}^{\infty} \frac{(-)^{m+1}}{m} \int d\,x_1 \ldots d\,x_m \, \boldsymbol{K}(x_1, x_2) \ldots \boldsymbol{K}(x_m, x_1) \qquad (1.102)$$

The necessary condition for the existence of $\ln\det \boldsymbol{M}$ is therefore that the trace of all powers of $\boldsymbol{K}$ exists.