2.2. Path Integral Representation: Special Hamiltonians

Real Time vs Imaginary Time

For convenience, we summarize the transition from real to imaginary time in 1-D space.
For clarity, real time is denoted by t & imaginary time by $\tau = i\,t$.

Reminder: Outside of this sub-section, the imaginary time is denoted by t.

Real time results:

$$L = \frac{1}{2} m\, q'^2 - V(q) \qquad\qquad q' = \frac{dq}{dt}$$

$$p = \frac{\partial L}{\partial q'} = m\, q'$$

$$H = \frac{\partial L}{\partial q'} q' - L = \frac{1}{2} m (q')^2 + V(q) = \frac{p^2}{2m} + V(q)$$

Action: $\mathbb{S} = \int_{t'}^{t''} dt\, L$

$$U(t'', t') = e^{i \mathbb{S}/\hbar} = \exp\left(\frac{i}{\hbar} \int_{t'}^{t''} dt\, L \right) = \exp\left[\frac{i}{\hbar} \int_{t'}^{t''} dt \left(\frac{1}{2} m\, q'^2 - V(q) \right) \right]$$

Euler-Lagrange eq. is $m\, q'' = -\dfrac{\partial V}{\partial q}$

Imaginary time version obtained from the real-time version by setting $t = -i\,\tau$:

$$L = -\frac{1}{2} m\, \dot{q}^2 - V(q) \qquad\qquad \dot{q} \equiv \frac{dq}{d\tau} = -i\, q'$$

$$p = -i \frac{\partial L}{\partial \dot{q}} = i\, m\, \dot{q}$$

$$H = \frac{\partial L}{\partial \dot{q}} \dot{q} - L = -\frac{1}{2} m\, \dot{q}^2 + V(q) = \frac{p^2}{2m} + V(q)$$

$$\mathbb{S} = -i \int_{\tau'}^{\tau''} d\tau\, L = i \int_{\tau'}^{\tau''} d\tau \left(\frac{1}{2} m\, \dot{q}^2 + V(q) \right)$$

→ $$U(\tau'', \tau') = \exp\left(\frac{1}{\hbar} \int_{\tau'}^{\tau''} d\tau\, L \right) = \exp\left[-\frac{1}{\hbar} \int_{\tau'}^{\tau''} d\tau \left(\frac{1}{2} m\, \dot{q}^2 + V(q) \right) \right]$$

Euler-Lagrange eq. is $m\, \ddot{q} = \dfrac{\partial V}{\partial q}$

Hereafter, we shall follow Zinn-Justin & use an “Euclidean action” S such that

$$U(\tau'', \tau') = e^{-S/\hbar}$$

→ $$S = i\, \mathbb{S} = -\int_{\tau'}^{\tau''} d\tau\, L = \int_{\tau'}^{\tau''} d\tau \left(\frac{1}{2} m\, \dot{q}^2 + V(q) \right)$$

Equation for U

Caution: As mentioned at the end of §2.1, t denotes the imaginary time.

Consider a system with

$$H = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V(\hat{\boldsymbol{q}}, t) \qquad\qquad [\hat{q}_\alpha, \hat{p}_\beta] = i\hbar\,\delta_{\alpha\beta}$$

(2.8)

$$\begin{aligned}\langle \boldsymbol{q} \,|\, [\hat{q}_\alpha, \hat{p}_\beta] \,|\, \boldsymbol{q}'\rangle &= \langle \boldsymbol{q} \,|\, (\hat{q}_\alpha \hat{p}_\beta - \hat{p}_\beta \hat{q}_\alpha) \,|\, \boldsymbol{q}'\rangle \\ &= \langle \boldsymbol{q} \,|\, (q_\alpha \hat{p}_\beta - \hat{p}_\beta q_\alpha') \,|\, \boldsymbol{q}'\rangle \\ &= (q_\alpha - q_\alpha')\langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle \\ &= i\hbar\,\delta_{\alpha\beta}\langle \boldsymbol{q} \,|\, \boldsymbol{q}'\rangle \qquad\qquad [\text{ from eq(2.8) }] \\ &= i\hbar\,\delta_{\alpha\beta}\,\delta(\boldsymbol{q} - \boldsymbol{q}')\end{aligned}$$

$$\therefore \qquad (q_\alpha - q_\alpha')\langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle = i\hbar\,\delta_{\alpha\beta}\,\delta(\boldsymbol{q} - \boldsymbol{q}')$$

$$\int d\boldsymbol{q}\,(q_\alpha - q_\alpha')\langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle = i\hbar\,\delta_{\alpha\beta}$$

Let

$$\langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle = a\frac{\partial}{\partial q_\beta}\delta(\boldsymbol{q} - \boldsymbol{q}') \qquad\qquad (\, a = \text{constant} \,)$$

then

$$\begin{aligned}\int d\boldsymbol{q}\,(q_\alpha - q_\alpha')\langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle &= a\int d\boldsymbol{q}\,(q_\alpha - q_\alpha')\frac{\partial}{\partial q_\beta}\delta(\boldsymbol{q} - \boldsymbol{q}') \\ &= a\left((q_\alpha - q_\alpha')\,\delta(\boldsymbol{q} - \boldsymbol{q}')\,\big|_{|\boldsymbol{q}| = \infty} - \int d\boldsymbol{q}\,\delta(\boldsymbol{q} - \boldsymbol{q}')\frac{\partial}{\partial q_\beta}(q_\alpha - q_\alpha')\right) \\ &= -a\,\delta_{\alpha\beta}\int d\boldsymbol{q}\,\delta(\boldsymbol{q} - \boldsymbol{q}') \\ &= -a\,\delta_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}\therefore \qquad \langle \boldsymbol{q} \,|\, \hat{p}_\beta \,|\, \boldsymbol{q}'\rangle &= -i\hbar\frac{\partial}{\partial q_\beta}\delta(\boldsymbol{q} - \boldsymbol{q}') \\ &= -i\hbar\frac{\partial}{\partial q_\beta}\langle \boldsymbol{q} \,|\, \boldsymbol{q}'\rangle\end{aligned}$$

$$\begin{aligned}\rightarrow \qquad \langle \boldsymbol{q} \,|\, H(t) \,|\, \boldsymbol{q}'\rangle &= \left[-\frac{\hbar^2}{2m}\nabla_{\boldsymbol{q}}^2 + V(\boldsymbol{q}, t)\right]\delta(\boldsymbol{q} - \boldsymbol{q}') \\ &= \left[-\frac{\hbar^2}{2m}\nabla_{\boldsymbol{q}}^2 + V(\boldsymbol{q}, t)\right]\langle \boldsymbol{q} \,|\, \boldsymbol{q}'\rangle\end{aligned}$$

The matrix elements of eq(2.5)

$$-\hbar\frac{\partial}{\partial t}U(t, t') = H\,U(t, t')$$

are

$$\begin{aligned}-\hbar\frac{\partial}{\partial t}\langle \boldsymbol{q} \,|\, U(t, t') \,|\, \boldsymbol{q}'\rangle &= \int d\boldsymbol{q}''\langle \boldsymbol{q} \,|\, H \,|\, \boldsymbol{q}''\rangle\langle \boldsymbol{q}'' \,|\, U(t, t') \,|\, \boldsymbol{q}'\rangle \\ &= \int d\boldsymbol{q}''\left[-\frac{\hbar^2}{2m}\nabla_{\boldsymbol{q}}^2 + V(\boldsymbol{q}, t)\right]\langle \boldsymbol{q} \,|\, \boldsymbol{q}''\rangle\langle \boldsymbol{q}'' \,|\, U(t, t') \,|\, \boldsymbol{q}'\rangle \\ &= \left[-\frac{\hbar^2}{2m}\nabla_{\boldsymbol{q}}^2 + V(\boldsymbol{q}, t)\right]\langle \boldsymbol{q} \,|\, U(t, t') \,|\, \boldsymbol{q}'\rangle\end{aligned}$$

(2.9)

Initial condition for eq(2.9) is

$$U(t', t') = 1$$

$$\to \quad \langle \boldsymbol{q} \mid U(t', t') \mid \boldsymbol{q}' \rangle = \langle \boldsymbol{q} \mid \boldsymbol{q}' \rangle = \delta(\boldsymbol{q} - \boldsymbol{q}')$$

(2.9a)

For $V = 0$, eq(2.9) reduces to

$$\left(-\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_{\boldsymbol{q}}^2 \right) \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = 0$$

Fourier Transform

For an d-D system, consider the Fourier transform

$$\langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{d\omega}{2\pi} U(\boldsymbol{p}, \omega) \exp\left[\frac{i}{\hbar} \boldsymbol{p} \cdot (\boldsymbol{q} - \boldsymbol{q}') - i\omega (t - t') \right]$$

$$\to \quad \left(-\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla_{\boldsymbol{q}}^2 \right) \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle$$

$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{d\omega}{2\pi} \left(i\hbar\omega - \frac{1}{2m} \boldsymbol{p}^2 \right) U(\boldsymbol{p}, \omega) \exp\left[\frac{i}{\hbar} \boldsymbol{p} \cdot (\boldsymbol{q} - \boldsymbol{q}') - i\omega (t - t') \right]$$

$$= 0$$

$$\therefore \quad \omega = -\frac{i}{2m\hbar} \boldsymbol{p}^2 \quad \& \quad U(\boldsymbol{p}, \omega) = \delta\left(\omega + \frac{i}{2m\hbar} \boldsymbol{p}^2 \right) U(\boldsymbol{p})$$

$$\to \quad \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{U(\boldsymbol{p})}{2\pi} \exp\left[\frac{i}{\hbar} \boldsymbol{p} \cdot (\boldsymbol{q} - \boldsymbol{q}') - \frac{\boldsymbol{p}^2}{2m\hbar} (t - t') \right]$$

Eq(2.9a) gives

$$\delta(\boldsymbol{q} - \boldsymbol{q}') = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{U(\boldsymbol{p})}{2\pi} \exp\left[\frac{i}{\hbar} \boldsymbol{p} \cdot (\boldsymbol{q} - \boldsymbol{q}') \right]$$

$$\to \quad \frac{U(\boldsymbol{p})}{2\pi} = 1$$

$$\therefore \quad \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \exp\left[\frac{i}{\hbar} \boldsymbol{p} \cdot (\boldsymbol{q} - \boldsymbol{q}') - \frac{\boldsymbol{p}^2}{2m\hbar} (t - t') \right]$$

In order to use the Gaussian integral formula eq(1.8) for commuting variables

$$\int d^n x \exp\left(-\frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} \right) = \frac{(2\pi)^{n/2}}{\sqrt{\det \boldsymbol{A}}} \exp\left(\frac{1}{2} \boldsymbol{b}^T \boldsymbol{A}^{-1} \boldsymbol{b} \right)$$

we set

$$\boldsymbol{A} = \operatorname{diag}\left(\frac{t - t'}{m\hbar} \right) \qquad \boldsymbol{b} = -\frac{i}{\hbar} (\boldsymbol{q} - \boldsymbol{q}')$$

$$\to \quad \det \boldsymbol{A} = \left(\frac{t - t'}{m\hbar} \right)^d \qquad \boldsymbol{A}^{-1} = \operatorname{diag}\left(\frac{m\hbar}{t - t'} \right)$$

$$\therefore \quad \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \left(\frac{m}{2\pi\hbar(t - t')} \right)^{d/2} \exp\left[-\frac{m}{2\hbar(t - t')} (\boldsymbol{q} - \boldsymbol{q}')^2 \right] \qquad (2.10)$$

Small Time Interval Solution

To solve eq(2.9) for small $\varepsilon = t - t'$, we set

$$\langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \exp\left[-\frac{1}{\hbar} \sigma(\boldsymbol{q}, \boldsymbol{q}'; t, t') \right]$$

so that

$$-\hbar \frac{\partial}{\partial t} \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \frac{\partial \sigma}{\partial t} \exp\left[-\frac{1}{\hbar} \sigma(\boldsymbol{q}, \boldsymbol{q}'; t, t') \right]$$

$$\nabla_{\boldsymbol{q}} \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = -\frac{1}{\hbar} \left(\nabla_{\boldsymbol{q}} \sigma \right) \exp\left[-\frac{1}{\hbar} \sigma(\boldsymbol{q}, \boldsymbol{q}'; t, t') \right]$$

$$\nabla_{\boldsymbol{q}}^2 \langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle = \left[-\frac{1}{\hbar} \nabla_{\boldsymbol{q}}^2 \sigma + \frac{1}{\hbar^2} \left(\nabla_{\boldsymbol{q}} \sigma \right)^2 \right] \exp\left[-\frac{1}{\hbar} \sigma(\boldsymbol{q}, \boldsymbol{q}'; t, t') \right]$$

and eq(2.9) becomes

$$\frac{\partial \sigma}{\partial t} = \frac{\hbar}{2m} \nabla_{\boldsymbol{q}}^2 \sigma - \frac{1}{2m} \left(\nabla_{\boldsymbol{q}} \sigma \right)^2 + V(\boldsymbol{q}, t)$$

(2.10a)

Assuming σ to be dominated by the free ($V = 0$) term, we first re-write eq(2.10) as

$$\langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle_{V=0} = \exp\left[-\frac{m}{2\hbar (t-t')} (\boldsymbol{q} - \boldsymbol{q}')^2 + \frac{d}{2} \ln\left(\frac{m}{2\pi\hbar (t-t')} \right) \right]$$

$$\equiv \exp\left(-\frac{1}{\hbar} \sigma_0 \right)$$

& set

$$\sigma(\boldsymbol{q}, \boldsymbol{q}'; \varepsilon) = \sigma_0 + \sigma_1$$

$$= \frac{m}{2\varepsilon} (\boldsymbol{q} - \boldsymbol{q}')^2 - \frac{d}{2} \hbar \ln\left(\frac{m}{2\pi\hbar\varepsilon} \right) + \sigma_1$$

$$= \frac{m}{2\varepsilon} (\boldsymbol{q} - \boldsymbol{q}')^2 + \frac{d}{2} \hbar \ln\left(\frac{2\pi\hbar\varepsilon}{m} \right) + \sigma_1$$

(2.11)

where $\sigma_1(\boldsymbol{q}, \boldsymbol{q}'; \varepsilon) = O(\varepsilon)$ is the 1st order correction due to V.

Using

$$\frac{\partial \sigma_0}{\partial t} = \frac{\hbar}{2m} \nabla_{\boldsymbol{q}}^2 \sigma_0 - \frac{1}{2m} \left(\nabla_{\boldsymbol{q}} \sigma_0 \right)^2$$

$$\nabla_{\boldsymbol{q}} \sigma_0 = \frac{m}{\varepsilon} (\boldsymbol{q} - \boldsymbol{q}')$$

& $\quad \left(\nabla_{\boldsymbol{q}} \sigma \right)^2 \approx \left(\nabla_{\boldsymbol{q}} \sigma_0 \right)^2 + 2 \left(\nabla_{\boldsymbol{q}} \sigma_0 \right) \cdot \left(\nabla_{\boldsymbol{q}} \sigma_1 \right) + O\left(\varepsilon^2 \right)$

we can rewrite eq(2.10a) to $O(\varepsilon)$ as

$$\frac{\partial \sigma_1}{\partial t} \approx \frac{\hbar}{2m} \nabla_{\boldsymbol{q}}^2 \sigma_1 - \frac{1}{m} \left(\nabla_{\boldsymbol{q}} \sigma_0 \right) \cdot \left(\nabla_{\boldsymbol{q}} \sigma_1 \right) + V(\boldsymbol{q}, t)$$

$$= \frac{\hbar}{2m} \nabla_{\boldsymbol{q}}^2 \sigma_1 - \frac{1}{\varepsilon} (\boldsymbol{q} - \boldsymbol{q}') \cdot \left(\nabla_{\boldsymbol{q}} \sigma_1 \right) + V(\boldsymbol{q}, t)$$

$$\rightarrow \quad \varepsilon \frac{\partial \sigma_1}{\partial t} + (\boldsymbol{q} - \boldsymbol{q}') \cdot \left(\nabla_{\boldsymbol{q}} \sigma_1 \right) = \varepsilon V(\boldsymbol{q}, t) \qquad (2.12a)$$

where the term $\varepsilon \nabla_{\boldsymbol{q}}^2 \sigma_1 = O\left(\varepsilon^2 \right)$ is dropped.

Note that $\varepsilon \frac{\partial \sigma_1}{\partial t} \approx \varepsilon \frac{\Delta \sigma_1}{\varepsilon} = O(\varepsilon)$.

For convenience, eq(2.12a) can be written as

$$\left[(t - t') \frac{\partial}{\partial t} + (\boldsymbol{q} - \boldsymbol{q}') \cdot \nabla_{\boldsymbol{q}} \right] \sigma_1 = (t - t') V(\boldsymbol{q}, t) \qquad (2.12)$$

or

$$\left[\frac{\partial}{\partial t}+\left(\frac{\boldsymbol{q}-\boldsymbol{q}'}{t-t'}\right)\cdot\nabla_{\boldsymbol{q}}\right]\sigma_1 = V(\boldsymbol{q},t)$$

Assuming a constant velocity, the trajectory in going from $\boldsymbol{q}'$ at t' to $\boldsymbol{q}$ at t is simply

$$\boldsymbol{q}(\tau)=\boldsymbol{q}'+\frac{\boldsymbol{q}-\boldsymbol{q}'}{t-t'}(\tau-t') \qquad \text{where} \qquad t\geq\tau\geq t'$$

(2.13)

$$\rightarrow \qquad \frac{d\,q(\tau)}{d\,\tau}=\frac{\boldsymbol{q}-\boldsymbol{q}'}{t-t'}$$

Since eq(2.13) is valid for $t\geq\tau\geq t'$, eq(2.12) can be written as

$$\left[\frac{\partial}{\partial t}+\frac{d\,q(t)}{d\,t}\cdot\nabla_{\boldsymbol{q}}\right]\sigma_1 = V(\boldsymbol{q},t)$$

$$=\frac{d\,\sigma_1}{d\,t}$$

Hence,

$$\sigma_1=\int_{t'}^{t} d\,\tau\, V[\boldsymbol{q}(\tau),\tau] \qquad (2.14)$$

Approximation eq(2.13) also leads to

$$\int_{t'}^{t} d\,\tau\left(\frac{d\,\boldsymbol{q}(\tau)}{d\,\tau}\right)^2 \approx \int_{t'}^{t} d\,\tau\left(\frac{\boldsymbol{q}-\boldsymbol{q}'}{t-t'}\right)^2$$

$$=\frac{(\boldsymbol{q}-\boldsymbol{q}')^2}{t-t'}$$

Eq(2.11) thus becomes

$$\sigma\approx\int_{t'}^{t} d\,\tau\left\{\frac{1}{2}m\left(\frac{d\,\boldsymbol{q}(\tau)}{d\,\tau}\right)^2+V[\boldsymbol{q}(\tau),\tau]\right\}+\frac{d}{2}\hbar\ln\left(\frac{2\pi\hbar\varepsilon}{m}\right)$$

so that

$$\langle\boldsymbol{q}\mid U(t,t')\mid\boldsymbol{q}'\rangle\approx\left(\frac{m}{2\pi\hbar\varepsilon}\right)^{d/2}\exp\left(-\frac{1}{\hbar}\int_{t'}^{t} d\,\tau\left\{\frac{1}{2}m\left(\frac{d\,\boldsymbol{q}(\tau)}{d\,\tau}\right)^2+V[\boldsymbol{q}(\tau),\tau]\right\}\right) \qquad (2.15)$$

where the integrand is correct to $O(\varepsilon)$.

Finite Time Interval

For the evolution of finite intervals, each specific path from $\boldsymbol{q}'$ to $\boldsymbol{q}''$ can be evaluated by dividing the interval into $n\rightarrow\infty$ infinitesimal segments to give

$$\lim_{n\rightarrow\infty}\prod_{k=1}^{n-1}\langle\boldsymbol{q}_{k+1}\mid U(t_{k+1},t_k)\mid\boldsymbol{q}_k\rangle=\lim_{n\rightarrow\infty}\left(\frac{m}{2\pi\hbar\varepsilon}\right)^{n\,d/2}\exp\left(-\frac{1}{\hbar}S[\boldsymbol{q},\varepsilon]\right)$$

where the functional

$$S[\boldsymbol{q},\varepsilon]=\int_{t'}^{t''} d\,t\left\{\frac{1}{2}m\left(\frac{d\,\boldsymbol{q}(t)}{d\,t}\right)^2+V[\boldsymbol{q}(t),t]\right\}+O\left(n\,\varepsilon^2\right)$$

(2.17)

is to be evaluated along the specified path $\boldsymbol{q}(t)$.

By definition, $\langle\boldsymbol{q}''\mid U(t,t')\mid\boldsymbol{q}'\rangle$ is the probability amplitude of the transition from $\mid\boldsymbol{q}'(t')\rangle$ to $\mid\boldsymbol{q}''(t'')\rangle$. Additivity of probability amplitudes means that we therefore need to sum over all possible paths to get

$$\langle \boldsymbol{q}'' \mid U(t'', t') \mid \boldsymbol{q}' \rangle_\varepsilon = \lim_{n\to\infty} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{n d/2} \int \prod_{k=1}^{n-1} d^d q_k \, \mu(\boldsymbol{q}) \exp\left(-\frac{1}{\hbar} S[\boldsymbol{q}, \varepsilon]\right) \qquad (2.16)$$

where $\mu(\boldsymbol{q})$ is the itegration measure that converts the sum into integrals.

Finding the Measure μ

The measure $\mu(\boldsymbol{q})$ can be determined by evaluating eq(2.19) explicitly for the 1-D free case & comparing the result with eq(2.10). To begin, we assume the simplest possibility & set $\mu =$ constant.

$$\langle \boldsymbol{q}'' \mid U(t'', t') \mid \boldsymbol{q}' \rangle = \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{n/2} \mu \prod_{k=1}^{n-1} \left\{ \int_{-\infty}^{\infty} d\, q_k \exp\left[-\frac{1}{2\hbar\varepsilon} m (q_k - q_{k-1})^2 \right]\right\}$$

The Gaussian integrals can be evaluated successively using

$$\int_{-\infty}^{\infty} d\, x \exp\left\{-\frac{1}{2} a\left[(q_1 - x)^2 + p\,(x - q_0)^2\right]\right\} = \sqrt{\frac{2\pi}{a\,(1+p)}} \exp\left[-\frac{a\,p}{2\,(1+p)} (q_1 - q_0)^2\right]$$

Starting with $k = 1$, we have

$$\sqrt{\frac{m}{2\pi\hbar\varepsilon}} \int_{-\infty}^{\infty} d\, q_1 \exp\left\{-\frac{m}{2\hbar\varepsilon}\left[(q_2 - q_1)^2 + (q_1 - q_0)^2\right]\right\} = \sqrt{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{m}{2\hbar\varepsilon}\right)(q_2 - q_0)^2\right]$$

The integral for $k = 2$ then becomes

$$\sqrt{\frac{m}{2\pi\hbar\varepsilon}} \int_{-\infty}^{\infty} d\, q_2 \exp\left\{-\frac{1}{2\hbar\varepsilon} m\left[(q_3 - q_2)^2 + \frac{1}{2}(q_2 - q_0)^2\right]\right\} = \sqrt{\frac{2}{3}} \exp\left[-\frac{1}{2}\left(\frac{m}{3\hbar\varepsilon}\right)(q_3 - q_0)^2\right]$$

The integral for $k = 3$ is

$$\sqrt{\frac{m}{2\pi\hbar\varepsilon}} \int_{-\infty}^{\infty} d\, q_3 \exp\left\{-\frac{1}{2\hbar\varepsilon} m\left[(q_4 - q_3)^2 + \frac{1}{3}(q_3 - q_0)^2\right]\right\} = \sqrt{\frac{3}{4}} \exp\left[-\frac{1}{2}\left(\frac{m}{4\hbar\varepsilon}\right)(q_4 - q_0)^2\right]$$

The integral for the general k is therefore

$$\sqrt{\frac{m}{2\pi\hbar\varepsilon}} \int_{-\infty}^{\infty} d\, q_k \exp\left\{-\frac{1}{2\hbar\varepsilon} m\left[(q_{k+1} - q_k)^2 + \frac{1}{k}(q_k - q_{k-1})^2\right]\right\}$$

$$= \sqrt{\frac{k}{k+1}} \exp\left[-\frac{m}{2\,(k+1)\,\varepsilon\,\hbar}(q_{k+1} - q_0)^2\right]$$

Since there're n prefactors $\sqrt{\frac{m}{2\pi\hbar\varepsilon}}$ but only $n - 1$ integrals, we have

$$\langle \boldsymbol{q}'' \mid U(t'', t') \mid \boldsymbol{q}' \rangle = \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \mu \sqrt{\frac{m}{2\pi\hbar\varepsilon}} \sqrt{\frac{1}{2}\,\frac{2}{3} \cdots \frac{n-2}{n-1}\,\frac{n-1}{n}} \exp\left[-\frac{m}{2\,n\,\varepsilon\,\hbar}(q_n - q_0)^2\right]$$

$$= \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \mu \sqrt{\frac{m}{2\pi\hbar\varepsilon}} \sqrt{\frac{1}{n}} \exp\left[-\frac{m}{2\,n\,\varepsilon\,\hbar}(q_n - q_0)^2\right]$$

$$= \mu \sqrt{\frac{m}{2\pi\hbar(t''-t')}} \exp\left[-\frac{m}{2(t''-t')\hbar}(q''-q')^2\right]$$

where

$n\varepsilon = t''-t' \qquad q_n = q'' \qquad q_0 = q'$

Comparing this with eq(2.10), we have $\mu = 1$.

Path Integral Form of U

In the limit $\varepsilon \to 0$, we have

$$S[\boldsymbol{q}] = \lim_{\varepsilon\to 0} S[\boldsymbol{q}, \varepsilon]$$

$$= \int_{t'}^{t''} dt \left\{\frac{1}{2} m \left(\frac{d\boldsymbol{q}(t)}{dt}\right)^2 + V[\boldsymbol{q}(t), t]\right\} \tag{2.18}$$

$$= -\int_{t'}^{t''} dt\, L$$

Eq(2.16) then becomes

$$\langle \boldsymbol{q}'' \mid U(t'', t') \mid \boldsymbol{q}' \rangle = \lim_{\substack{n\to\infty \\ \varepsilon\to 0}} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{nd/2} \int \prod_{k=1}^{n-1} d^d q_k \exp\left(-\frac{1}{\hbar} S[\boldsymbol{q}, \varepsilon]\right)$$

$$\equiv \int_{\boldsymbol{q}(t')=q'}^{\boldsymbol{q}(t'')=q''} [d\,\boldsymbol{q}(t)] \exp\left(-\frac{1}{\hbar} S[\boldsymbol{q}]\right)$$

(2.19)

where the last express defines the (Feymann) path integral of $\exp\left(-\frac{1}{\hbar} S[\boldsymbol{q}]\right)$ from $\boldsymbol{q}'(t')$ to $\boldsymbol{q}''(t'')$.

Note that the normalization factor

$$\mathcal{N} = \lim_{n\to\infty} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{nd/2} \tag{2.20}$$

is absorbed into the symbol $[d\,\boldsymbol{q}(t)]$. Its value can be derived by considering the free ($V=0$) case.

Fundamental Remarks

The most singular term in σ of eq(2.11) when $\varepsilon\to 0$ is $\frac{m}{2\varepsilon}(\boldsymbol{q}-\boldsymbol{q}')^2$. In order for this term to be finite, we must have $|\boldsymbol{q}-\boldsymbol{q}'| = O\left(\sqrt{\varepsilon}\right)$, irregardless of the potential V. The kinetic energy term is therefore part of the functional measure and determines the functional space over which to integrate.

For the same reason, the support of $\langle \boldsymbol{q} \mid U(t, t') \mid \boldsymbol{q}' \rangle$, i.e., region in which $U(t, t') \neq 0$, is $|\boldsymbol{q}-\boldsymbol{q}'| = O\left(\sqrt{\varepsilon}\right)$. Hence, eq(2.14) becomes

$$\sigma_1(\boldsymbol{q}, \boldsymbol{q}'; \varepsilon) = \varepsilon V\left(\frac{\boldsymbol{q}+\boldsymbol{q}'}{2}, t'\right) + O(\varepsilon^2)$$

$$V\left(\frac{\boldsymbol{q}+\boldsymbol{q}'}{2}\right) = V(\boldsymbol{q}) + \frac{1}{2}(\boldsymbol{q}'-\boldsymbol{q})\cdot\nabla_{\boldsymbol{q}} V(\boldsymbol{q}) + O\left(|\boldsymbol{q}-\boldsymbol{q}'|^2\right)$$

$$= V(\boldsymbol{q}') + \frac{1}{2}(\boldsymbol{q}-\boldsymbol{q}')\cdot\nabla_{\boldsymbol{q}'} V(\boldsymbol{q}') + O\left(|\boldsymbol{q}-\boldsymbol{q}'|^2\right)$$

$$= \frac{1}{2}[\,V(\boldsymbol{q}) + V(\boldsymbol{q}')\,] + O(\,|\,\boldsymbol{q} - \boldsymbol{q}'\,|\,)$$

$$= \frac{1}{2}[\,V(\boldsymbol{q}) + V(\boldsymbol{q}')\,] + O\left(\sqrt{\varepsilon}\right)$$

$$\therefore \qquad \sigma_1(\boldsymbol{q}, \boldsymbol{q}'; \varepsilon) = \varepsilon\,\frac{1}{2}[\,V(\boldsymbol{q}) + V(\boldsymbol{q}')\,] + O\left(\varepsilon^{3/2}\right)$$

This means any modification of the terms in σ of eq(2.11) can be dropped if it’s of $O\left(\varepsilon^{3/2}\right)$.

The relevant paths are typical of the brownian motion which are continuous, but not differentiable. Foregoing discussion on the support of the path integral suggests that the largest contribution must come from the classical (differentiable) Euclidean path given by

$$\frac{\delta S}{\delta \boldsymbol{q}} = 0 \;\; \& \qquad\qquad \frac{\delta^2 S}{\delta \boldsymbol{q}\, \delta \boldsymbol{q}} \geq 0$$

The semi-classical approximations are based on this observation.

Many Particle Systems

Generalization to many particle systems simply means inserting the appropriate Lagrangian / action in eq(2.18).