2.3. Explicit Evaluation of a Path Integral: The Harmonic Oscillator

For a 1-D harmonic oscillator,

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2 \qquad (2.21)$$

$$-L = \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2 \qquad (2.21a)$$

$$S_0[q] = -\int_{t'}^{t''} dt\, L = \int_{t'}^{t''} dt \left(\frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2 \right) \qquad (2.22)$$

Eq(2.19) becomes

$$\langle q'' \mid U_0(t'', t') \mid q' \rangle = \int_{q'}^{q''} [d\, q(t)] \exp\left(-\frac{1}{\hbar} S_0[q] \right) \qquad (2.23)$$

The Euler-Lagrange eq. for eq(2.21a) is

$$m \ddot{q} - m \omega^2 q = 0 \qquad (2.24a)$$

Let q_c be its solution satisfying the boundary conditions

$$q_c(t') = q' \qquad \& \qquad q_c(t'') = q'' \qquad (2.24)$$

To evaluate eq(2.23), we set

$$q(t) = q_c(t) + r(t) \qquad (2.24b)$$

At the limits of the integrals,

$$q(t') = q' \qquad \& \qquad q(t'') = q''$$

→ $$r(t') = r(t'') = 0 \qquad (2.25)$$

Now,

$$q^2 = q_c^2 + r^2 + 2\, q_c\, r$$

$$\dot{q} = \dot{q}_c + \dot{r}$$

$$\dot{q}^2 = \dot{q}_c^2 + \dot{r}^2 + 2\, \dot{q}_c\, \dot{r}$$

Eq(2.22) thus becomes

$$S_0[q] = S_0[q_c] + S_0[r] + m \int_{t'}^{t''} dt \left(\dot{q}_c\, \dot{r} + \omega^2 q_c\, r \right)$$

Using

$$\int_{t'}^{t''} dt\, \dot{q}_c\, \dot{r} = \int_{t'}^{t''} dr\, \dot{q}_c = \dot{q}_c\, r \Big|_{t'}^{t''} - \int_{t'}^{t''} r\, d\dot{q}_c$$

$$= -\int_{t'}^{t''} r\, \ddot{q}_c\, dt \qquad \text{[from eq(2.25)]}$$

→ $$\int_{t'}^{t''} dt \left(\dot{q}_c\, \dot{r} + \omega^2 q_c\, r \right) = \int_{t'}^{t''} dt \left(-\ddot{q}_c + \omega^2 q_c \right) r = 0 \qquad \text{[from eq(2.24a)]}$$

∴ $$S_0[q] = S_0[q_c] + S_0[r] \qquad (2.25a)$$

The Classical Action

The general solutions of eq(2.24a) are of the form

$$q(t) = A \sinh \omega t + B \cosh \omega t$$

The boundary conditions eq(2.24) becomes

$$q' = A \sinh \omega t' + B \cosh \omega t' \qquad \& \qquad q'' = A \sinh \omega t'' + B \cosh \omega t''$$

→ $$A = (q' \cosh \omega t'' - q'' \cosh \omega t') / (\cosh \omega t'' \sinh \omega t' - \cosh \omega t' \sinh \omega t'')$$

$$= \frac{-q' \cosh \omega t'' + q'' \cosh \omega t'}{\sinh \omega (t'' - t')}$$

$$B = (q' \sinh \omega t'' - q'' \sinh \omega t') / (\cosh \omega t' \sinh \omega t'' - \sinh \omega t' \cosh \omega t'')$$

$$= \frac{q' \sinh \omega t'' - q'' \sinh \omega t'}{\sinh \omega (t'' - t')}$$

$$\therefore \qquad q_c(t) = q' ((-\cosh \omega t'' \sinh \omega t + \sinh \omega t'' \cosh \omega t) / (\sinh \omega (t'' - t'))) + q'' \frac{\cosh \omega t' \sinh \omega t - \sinh \omega t' \cosh \omega t}{\sinh \omega (t'' - t')}$$

$$= \frac{1}{\sinh \omega (t'' - t')} [q' \sinh \omega (t'' - t) + q'' \sinh \omega (t - t')] \tag{2.26}$$

Using

$$\int_{t'}^{t''} dt\, \dot{q}^2 = \int_{t'}^{t''} dq\, \dot{q} = q \dot{q} \Big|_{t'}^{t''} - \int_{t'}^{t''} d\dot{q}\, q$$

$$= q \dot{q} \Big|_{t'}^{t''} - \int_{t'}^{t''} dt\, \ddot{q}\, q$$

& eq(2.24a), we have

$$\int_{t'}^{t''} dt\, \dot{q}_c^2 = q'' \dot{q}_c(t'') - q' \dot{q}_c(t') - \omega^2 \int_{t'}^{t''} dt\, q_c^2$$

so that

$$S_0[q_c] = \frac{1}{2} m [q'' \dot{q}_c(t'') - q' \dot{q}_c(t')]$$

From eq(2.26),

$$\dot{q}_c(t) = \frac{\omega}{\sinh \omega (t'' - t')} [-q' \cosh \omega (t'' - t) + q'' \cosh \omega (t - t')]$$

so that

$$S_0[q_c] = \frac{1}{2} m \left\{ q'' \frac{\omega}{\sinh \omega (t'' - t')} [-q' + q'' \cosh \omega (t'' - t')] - q' \frac{\omega}{\sinh \omega (t'' - t')} [-q' \cosh \omega (t'' - t') + q''] \right\}$$

$$= \frac{m \omega}{2 \sinh \omega (t'' - t')} \left\{ -2 q' q'' + \left[(q'')^2 + (q')^2 \right] \cosh \omega (t'' - t') \right\} \tag{2.27}$$

The Path Integral

Using eq(2.25a) on eq(2.23), we have

$$\langle q'' \mid U_0(t'', t') \mid q' \rangle = \exp\left(-\frac{1}{\hbar} S_0[q_c] \right) \mathcal{N}(\omega; t'' - t) \tag{2.28}$$

where

$$\mathcal{N}(\omega; t'' - t) = \int [d r(t)] \exp\left[-\frac{1}{\hbar} \int_{t'}^{t''} dt \left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \omega^2 r^2 \right) \right] \tag{2.29}$$

with eq(2.25)

$$r(t') = r(t'') = 0$$

Note that $\mathcal{N}(\omega;\, t'' - t)$ depends only on the length of the time interval $t'' - t$ so that all the spatial dependencies are contained in the action $S_0[q_c]$ on the classical path.

The evaluation of eq(2.29) is postphoned to §2.5.2. We simply quote the final result here

$$\mathcal{N} = \sqrt{\frac{m\,\omega}{2\,\pi\,\hbar \sinh \omega(t'' - t')}}$$

so that

$$\langle q'' \mid U_0(t'', t') \mid q' \rangle = \sqrt{\frac{m\,\omega}{2\,\pi\,\hbar \sinh \omega(t'' - t')}} \tag{2.30}$$

$$\times \exp\left(\frac{m\,\omega}{2 \sinh \omega\,(t'' - t')} \left\{ -2\,q'\,q'' + \left[(q'')^2 + (q')^2\right] \cosh \omega\,(t'' - t') \right\} \right)$$