3.1. General Hamiltonians: Phase Space Path Integral

Consider a local hamiltonian $\hat{H}(t)$. The imaginary time equation of motion for the evolution operator U is [see eq(2.5)]

$$\hbar \frac{\partial U(t, t')}{\partial t} = -\hat{H}(t)\, U(t, t') \qquad \text{with} \qquad U(t', t') = 1 \tag{3.1}$$

satisfied by eq(2.7)

$$\langle q'' \mid U(t'', t') \mid q' \rangle = \int \prod_{k=1}^{n-1} d\,q_k \prod_{k=1}^{n} \langle q_k \mid U(t_k, t_{k-1}) \mid q_{k-1} \rangle \tag{3.2}$$

where

$$t_k = t' + k\,\varepsilon \qquad t'' = t_n = t' + n\,\varepsilon \qquad q_0 = q' \qquad q_n = q''$$

In this chapter, we consider the quantization of a classical hamiltonian $H(p, q, t)$ while preserving its symmetries.

The Phase Space Path Integral

For $\varepsilon \to 0$, eq(3.1) becomes

$$\hbar \frac{U(t+\varepsilon, t') - U(t, t')}{\varepsilon} = -\hat{H}(t)\, U(t, t') + O(\varepsilon)$$

$$\to \qquad U(t+\varepsilon, t') = U(t, t')\left[1 - \frac{\varepsilon}{\hbar}\hat{H} + O\left(\varepsilon^2\right)\right]$$

Setting $t' = t$ gives

$$U(t+\varepsilon, t) = 1 - \frac{\varepsilon}{\hbar}\hat{H}(\hat{p}, \hat{q}, t) + O\left(\varepsilon^2\right)$$

Setting $t \to t - \varepsilon$ gives

$$U(t, t-\varepsilon) = 1 - \frac{\varepsilon}{\hbar}\hat{H}(\hat{p}, \hat{q}, t-\varepsilon) + O\left(\varepsilon^2\right)$$

$$= 1 - \frac{\varepsilon}{\hbar}\hat{H}(\hat{p}, \hat{q}, t) + O\left(\varepsilon^2\right) \tag{3.3}$$

We shall assume $\hat{H}$ to have a classical limit $H(p, q, t)$, i.e.,

$$\lim_{\substack{q' \to q \\ \hbar \to 0}} \left\langle q \left| \hat{H}(\hat{p}, \hat{q}, t) \right| q' \right\rangle = H(p, q, t)$$

$$\to \qquad \langle q \mid U(t, t-\varepsilon) \mid q' \rangle = \delta(q - q') - \frac{\varepsilon}{\hbar}\left\langle q \left| \hat{H}(\hat{p}, \hat{q}, t) \right| q' \right\rangle + O\left(\varepsilon^2\right) \tag{3.3a}$$

Since $\hat{p}$ & $\hat{q}$ do not commute, the quantized form $\hat{H}(\hat{p}, \hat{q}, t)$ of $H(p, q, t)$ is not unique if H is not separable. One way to get around this is to replace $\hat{q}$ with a function $q_{\text{av}}(q, q')$ so that

$$\left\langle q \left| \hat{H}(\hat{p}, \hat{q}, t) \right| q' \right\rangle = \left\langle q \left| \hat{H}(\hat{p}, q_{\text{av}}, t) \right| q' \right\rangle$$

$$= \int d\,p \left\langle q \left| \hat{H}(\hat{p}, q_{\text{av}}, t) \right| p \right\rangle \langle p \mid q' \rangle$$

$$= \int d\,p\; H(p, q_{\text{av}}, t) \langle q \mid p \rangle \langle p \mid q' \rangle$$

$$= \int \frac{d\,p}{2\pi\hbar} H(p, q_{\text{av}}, t)\, e^{i\,p\,(q-q')/\hbar} \tag{3.4}$$

where

$$\langle q \mid p \rangle \equiv \frac{e^{i p q/\hbar}}{\sqrt{2 \pi \hbar}}$$

is the q-representation of the eigenstate of $\hat{p}$ with eigenvalue p.

Specifying q_{av} is thus equivalent to setting the quantization rule.

For a hermitian $\hat{H}$, taking the complex conjugate of eq(3.4) gives

$$\left\langle q \left| \hat{H}(\hat{p}, \hat{q}, t) \right| q' \right\rangle^* = \int \frac{dp}{2\pi\hbar} H[p, q_{av}(q, q'), t]^* e^{-i p (q-q')/\hbar}$$

$$= \left\langle q' \left| \hat{H}(\hat{p}, \hat{q}, t) \right| q \right\rangle$$

$$= \int \frac{dp}{2\pi\hbar} H[p, q_{av}(q', q), t] e^{-i p (q-q')/\hbar}$$

→ $H[p, q_{av}(q, q'), t]^* = H[p, q_{av}(q', q), t]$

Hence, if we choose q_{av} to be a symmetry function, i.e.,

$$q_{av}(q, q') = q_{av}(q', q)$$

then the classical hamiltonian $H(p, q, t) \equiv H(p, q_{av}, t)$ is real.

One such example is the Wigner quantization rule

$$q_{av}(q, q') = \frac{1}{2}(q + q')$$

Using

$$\delta(q - q') = \int \frac{dp}{2\pi\hbar} e^{i p (q-q')/\hbar}$$

eq(3.3a) becomes, for $\varepsilon \to 0$,

$$\langle q \mid U(t, t-\varepsilon) \mid q' \rangle = \int \frac{dp}{2\pi\hbar} e^{i p (q-q')/\hbar} \left[1 - \frac{\varepsilon}{\hbar} H(p, q_{av}, t) \right] + O(\varepsilon^2)$$

$$= \int \frac{dp}{2\pi\hbar} e^{i p (q-q')/\hbar} \exp\left[-\frac{\varepsilon}{\hbar} H(p, q_{av}, t) \right] + O(\varepsilon^2)$$

$$= \int \frac{dp}{2\pi\hbar} \exp\left\{ \frac{1}{\hbar} [\, i p(q - q') - \varepsilon H(p, q_{av}, t) \,] \right\} + O(\varepsilon^2) \qquad (3.5)$$

For $q = q_k$, $q' = q_{k-1}$, we have

$$\dot{q}_k \approx \frac{q_k - q_{k-1}}{\varepsilon}$$

$$i p_k (q_k - q_{k-1}) - \varepsilon H[p, q_{av}(q_k, q_{k-1}), t] \approx \varepsilon \{ i p_k \dot{q}_k - H[p, q_{av}(q_k, q_{k-1}), t] \}$$

$$\approx \varepsilon L(q_k, \dot{q}_k, t)$$

∴ $$S_\varepsilon(p, q) = -\varepsilon \sum_{k=1}^{n} L(q_k, \dot{q}_k, t)$$

$$= \varepsilon \sum_{k=1}^{n} \{ -i p_k \dot{q}_k + H[p, q_{av}(q_k, q_{k-1}), t] \} \qquad (3.7)$$

Since the number of p_k's is n in eq(3.7), we have

$$\langle q'' \mid U(t'', t') \mid q' \rangle = \lim_{n \to \infty} \int \left(\prod_{k=1}^{n} \frac{d p_k}{2\pi\hbar} \right) \left(\prod_{k=1}^{n-1} d q_k \right) e^{-S_\varepsilon/\hbar}$$

$$= \lim_{n \to \infty} \int \prod_{k=1}^{n} \frac{d p_k \, d q_k}{2\pi\hbar} \Bigg) \delta(q_n - q'') e^{-S_\varepsilon/\hbar} \qquad (3.6)$$

with

$$q'' = q_n \qquad q' = q_0$$

Setting

$$p(t_k) = p_k \qquad q(t_k) = q_k$$

eqs(3.7 & 3.6) become

$$S(p, q) = \int_{t'}^{t''} d\,t \left\{ -i\, p(t)\, \dot{q}(t) + H[\, p(t),\, q(t),\, t\,] \right\} \tag{3.8}$$

$$\langle\, q'' \mid U(t'', t') \mid q' \,\rangle = \int_{q(t')=q'}^{q(t'')=q''} [d\, p(t)\, d\, q(t)\,]\, e^{-S(p,q)/\hbar} \tag{3.9}$$

where the path integral

$$\int_{q(t')=q'}^{q(t'')=q''} [d\, p(t)\, d\, q(t)\,] = \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{d\, p_k\, d\, q_k}{2\,\pi\,\hbar}$$

is just the integration in phase space for a field in classical statistics. Thus, it is invariant under a canonical transformation that preserves the Poisson bracket, i.e.,

$$(\,p,\; q\,) \to \;(P,\, Q) \qquad \text{such that} \qquad \{\,p,\, q\,\} = \{\,P,\; Q\,\}$$

See sub-sections “Remarks” & “The space of integration” in Zinn-Justin’s text for some useful pointers.