A3.1. Symplectic Form and Quantization: General Remarks

Ref: See for example, B.Schutz, "Geometrical Methods of Mathematical Physic", in particular Part B of Chapter 5.

Non-trivial topological properties of phase space, irrelevant from the point of view of classical mechanics, can affect quantization.

Consider the classical action in Hamiltonian formulation

$$\mathcal{A}(\boldsymbol{p}, \boldsymbol{q}) = \int_{t'}^{t''} d\,t \left\{ \boldsymbol{p}(t) \cdot \dot{\boldsymbol{q}}(t) - H[\,\boldsymbol{p}(t), \boldsymbol{q}(t), t\,] \right\} \tag{A3.1}$$

in which

$$\int_{t'}^{t''} d\,t\, \boldsymbol{p}(t) \cdot \dot{\boldsymbol{q}}(t) = \int_{\boldsymbol{q}(t')}^{\boldsymbol{q}(t'')} \boldsymbol{p} \cdot d\,\boldsymbol{q}$$

= Sum of areas in phase space between trajectory C & every $\boldsymbol{p} = 0$ axes

$$= \int_D d\,p_i \wedge d\,q^i \tag{A3.2}$$

where the boundary $\partial\, D$ of the domain D contains C & sections of all the $\boldsymbol{p} = 0$ axes.

Note that d in eq(A3.2) is the exterior derivative. Let f be any function (or 0-form), then $d\,f$, which equals to the ordinary differential of f, is called an 1-form. The defining characteristic of forms is their anti-symmetry, which in turn leads to the property $d^2 = 0$.

$$\omega = d\,p_i \wedge d\,q^i \tag{A3.2a}$$

is called a symplectic 2-form [see Schutz, §5.10].

Using $d^2 = 0$, we see that ω is closed, i.e.,

$$d\,\omega = d^2\,p_i \wedge d\,q^i - d\,p_i \wedge d^2\,q^i = 0 \tag{A3.5}$$

Any closed form is exact, i.e., there exists a 1-form ω' such that

$$\omega = d\,\omega' \tag{A3.3}$$

By definition,

$$d\,f = \frac{\partial f}{\partial\, q^j}\, d\,q^j$$

for any function $f = f(q)$.

Let

$$\omega' = \alpha_i(q)\, d\,q^i$$

we have

$$d\,\omega' = \frac{\partial\, \alpha_i}{\partial\, q^j}\, d\,q^j \wedge d\,q^i$$

Thus, setting

$$\frac{\partial\, \alpha_i}{\partial\, q^j}\, d\,q^j = d\,p_i \tag{A3.3a}$$

recovers eq(A3.3).

Using eqs(A3.2 & A3.2a), we can write eq(A3.1) as

$$\mathcal{A}(\boldsymbol{p}, \boldsymbol{q}) = \int_D \omega - \int_{t'}^{t''} d\,t\, H[\,\boldsymbol{p}(t), \boldsymbol{q}(t), t\,] \tag{A3.6a}$$

Let $\boldsymbol{u}$ be an $2\,n$-D vector whose components can be separated into 2 groups related by the analog of eq(A3.3a). The transformation $\{\boldsymbol{q}, \boldsymbol{p}\} \rightarrow \{\,\boldsymbol{u}\,\}$ is then called canonical. Eq(A3.2a) can be replaced by

$$\omega = \omega_{ij}\, d\,u_i \wedge d\,u_j$$

& eq(A3.6a) becomes

$$\mathcal{A}(\boldsymbol{u}) = \int_D \omega - \int_{t'}^{t''} d\,t\, H[\,\boldsymbol{u}(t),\, t\,] \qquad \text{(A3.6)}$$

$$= \omega'\Big|_{\partial D} - \int_{t'}^{t''} d\,t\, H[\,\boldsymbol{u}(t),\, t\,]$$

Consider

$$\mathcal{I} = \int_{t'}^{t''} d\,t\, p_i\, \dot{q}^i = \int_D d\,p_i \wedge d\,q^i = \int_D \omega$$

$$= \int_D \omega_{ij}\, d\,u_i \wedge d\,u_j$$

Allowing ω_{ij} to sort out the assignment of the the role “coordinates” & “momenta”, we have

$$\mathcal{I} = \int_{t'}^{t''} d\,t\;\; \omega_{ij}\, u_i\, \dot{u}_j$$

The equations of motion is obtained by extremizing the integrand of $\mathcal{A}$ with respect to $\boldsymbol{u}$. For u_i, we have

$$\omega_{ij}\, \dot{u}_j - \frac{\partial H}{\partial u_i} = 0 \qquad \text{(A3.7)}$$

Example

The 2-D phase space of an 1-D system can be treated as a complex plane parametrized by the variables z & z^*. A common choice for the transformation $\{q, p\} \to \{z, z^*\}$ is

$$z = \frac{1}{\sqrt{2}}(\,q + i\,p\,) \qquad\qquad z^* = \frac{1}{\sqrt{2}}(\,q - i\,p\,)$$

where the prefactor $\frac{1}{\sqrt{2}}$ has no dynamical significance & is added only to make the inverse transformation

$$q = \frac{1}{\sqrt{2}}(z + z^*) \qquad\qquad p = \frac{1}{i\sqrt{2}}(z - z^*)$$

looks similar.

Using

$$d\,q = \frac{1}{\sqrt{2}}(d\,z + d\,z^*) \qquad\qquad d\,p = \frac{1}{i\sqrt{2}}(d\,z - d\,z^*)$$

and

$$d\,f \wedge d\,g \equiv -d\,g \wedge d\,f \quad \to \qquad d\,f \wedge d\,f = 0$$

we have

$$d\,p \wedge d\,q = \frac{1}{i}\, d\,z \wedge d\,z^*$$