A3.2. Spin Dynamics and Quantization

A3.2.1. Classical Spin Dynamics

Consider a 3-D vector $\boldsymbol{S}$ of length s, i.e.,

$$\boldsymbol{S}^2 = s^2$$

If s is fixed, we have

$$2\,\boldsymbol{S}\cdot\frac{d\,\boldsymbol{S}}{d\,t} = 2\,s\,\frac{d\,s}{d\,t} = 0$$

The simplest dynamics we can write for $\boldsymbol{S}$ is

$$\frac{d\,\boldsymbol{S}}{d\,t} = \boldsymbol{S}\times\boldsymbol{H} \qquad \text{(A3.8a)}$$

where $\boldsymbol{H}$ is a constant vector. The fixed length condition is automatically satisfied since

$$\boldsymbol{S}\cdot\frac{d\,\boldsymbol{S}}{d\,t} = \boldsymbol{S}\cdot(\boldsymbol{S}\times\boldsymbol{H}) = 0 \qquad \forall\ \boldsymbol{S}$$

Note: eq(A3.8a) differs from Zinn-Justin's eq(3.8) by a negative sign. We've chosen eq(A.3.8a) to conform with the case when $\boldsymbol{S}$ is an angular momentum with a gyromagnetic ratio $g = +1$ & $\boldsymbol{H}$ a magnetic field so that

$$\boldsymbol{F} = \frac{d\,\boldsymbol{S}}{d\,t} = \boldsymbol{\Gamma} = \boldsymbol{M}\times\boldsymbol{H} = g\,\boldsymbol{S}\times\boldsymbol{H}$$

where $\boldsymbol{F}$ = force, $\boldsymbol{\Gamma}$ = torque, & $\boldsymbol{M}$ = magnetic moment.
As in the case of the magnetic hamiltonian, Zinn-Justin's choice describes electrons.

Since $\boldsymbol{S}$ has only 2 independent components, eq(A3.8) is a set of 1st order differential eqs of 2 (angular) variables. Treated as eqs of motion in the hamiltonian formulism, these 2 variables become a conjugate pair of coordinate & momentum.

A natural choice is

$$\boldsymbol{S} = s\,(\,\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\,) = \boldsymbol{S}(\theta,\ \varphi) \qquad \text{(A3.8b)}$$

so that

$$\dot{\boldsymbol{S}} = s\left(\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi,\ \dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi,\ -\dot\theta\sin\theta\right)$$

Without lost of generality, we can orient the coordinates so that $\boldsymbol{H} = H\,\hat{\boldsymbol{z}}$. In which case,

$$\boldsymbol{S}\cdot\boldsymbol{H} = \cos\theta$$

$$\boldsymbol{S}\times\boldsymbol{H} = s\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & 0 & H \end{vmatrix}$$

$$= s\,H\,(\,\sin\theta\sin\varphi\,,\ -\sin\theta\cos\varphi,\ 0\,)$$

Eq(A3.8) becomes

$$\dot\theta\cos\theta\cos\varphi - \dot\varphi\sin\theta\sin\varphi = H\sin\theta\sin\varphi$$

$$\dot\theta\cos\theta\sin\varphi + \dot\varphi\sin\theta\cos\varphi = -H\sin\theta\cos\varphi$$

$$\dot\theta\sin\theta = 0$$

$$\rightarrow \qquad \dot\theta = 0 \qquad \& \qquad \dot\varphi = -H$$

The hamiltonian $\mathcal{H}$ must be a scalar so that the simplest form is (with $g = +1$)

$$\mathcal{H} = -\boldsymbol{M}\cdot\boldsymbol{H} = -g\,\boldsymbol{S}\cdot\boldsymbol{H} = -\boldsymbol{S}\cdot\boldsymbol{H} \qquad \text{(A3.9a)}$$

$$= -s\,H\cos\theta$$

Note the sign difference between eq(A.39a) & Zinn-Justin's eq(A3.9).

Choosing φ as the coordinate, its conjugate momentum p satisfies

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial \varphi} = 0$$

Since $\dot{\theta} = 0$, we may set

$$p = f(\theta) \qquad\qquad (f = \text{arbitrary function })$$

In order to satisfy

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial p} = -H$$

we have

$$p = s \cos\theta$$

so that

$$\mathcal{H} = -p H$$

Note that the coordinate-momentum conjugate pair is $(q, p) = (\varphi,\ s\cos\theta)$.

The Lagragian is therefore

$$\mathcal{L} = p\,\dot{\varphi} - \mathcal{H} = s\cos\theta\,\dot{\varphi} - \mathcal{H}$$

and the action

$$\mathcal{A} = \int d\,t\,\mathcal{L} = \int d\,t \left(s\cos\theta\,\dot{\varphi} - \mathcal{H} \right) \qquad \text{(A3.10)}$$

Note that using spherical coordinates $(\theta,\ \phi)$ makes the z-axis singular since ϕ cannot be specified there. Any integral that crosses the z-axis is therefore ill-defined.

For example, when one integrates the (symplectic) 2-form $d\,\varphi \wedge d\cos\theta$ to get the area of the sphere, one correct way is to pick a longitude, remove both poles from it, & then measure the area it sweeps out when rotated about the z-axis. In other words, integrate $\cos\theta$ first, then φ.

However, although increasing φ by $2\,\pi\,n$ returns the longitude to its original position, the integral gives $4\,\pi\,n$. The area thus obtained is therefore only defined $\bmod 4\,\pi$.

In classical mechanics, these (topological) properties are irrelevant since only equations of motion are physical.

The symplectic form has other useful representations.

From eq(A3.8b), we have

$$\cos\theta = \frac{S_3}{s} \qquad\qquad \varphi = \tan^{-1}\frac{S_2}{S_1}$$

$$\rightarrow \qquad d\cos\theta = \frac{s\,d\,S_3 - S_3\,d\,s}{s^2} \qquad\qquad d\,\varphi = \frac{S_1\,d\,S_2 - S_2\,d\,S_1}{S_1^2 + S_2^2}$$

$$s^2 = S_1^2 + S_2^2 + S_2^2$$

$$\rightarrow \qquad s\,d\,s = S_1\,d\,S_1 + S_2\,d\,S_2 + S_3\,d\,S_3$$

$$\therefore \qquad S_3\,d\,s = \frac{S_3}{s}\left(S_1\,d\,S_1 + S_2\,d\,S_2 + S_3\,d\,S_3 \right)$$

$$d\cos\theta = \frac{S_3}{s^3}\left[-S_1\,d\,S_1 - S_2\,d\,S_2 + \left(\frac{s^2}{S_3} - S_3 \right) d\,S_3 \right]$$

$$= \frac{S_3}{s^3}\left(-S_1\,d\,S_1 - S_2\,d\,S_2 + \frac{S_1^2 + S_2^2}{S_3}\,d\,S_3 \right)$$

By definition, if A, B are functions, then

$$d\,A \wedge d\,B = -d\,B \wedge d\,A \qquad\qquad d\,A \wedge d\,A = 0$$

$$\to \quad d\varphi\wedge d\cos\theta = \frac{S_3}{s^3(S_1^2+S_2^2)}\left[\,(S_1^2+S_2^2)\,dS_1\wedge dS_2 + \frac{S_1^2+S_2^2}{S_3}\,(S_1\,dS_2\wedge dS_3 + S_2\,dS_3\wedge dS_1)\right]$$

$$= \frac{1}{s^3}\,(S_3\,dS_1\wedge dS_2 + S_1\,dS_2\wedge dS_3 + S_2\,dS_3\wedge dS_1)$$

$$= \frac{1}{2}\,s^{-3}\,\varepsilon_{ijk}\,S_i\,dS_j\wedge dS_k$$

where the $\frac{1}{2}$ factor is necessary since, for example,

$$\varepsilon_{1jk}\,S_1\,dS_j\wedge dS_k = S_1(\,dS_2\wedge dS_3 - dS_3\wedge dS_2)$$

$$= 2\,S_1\,dS_2\wedge dS_3$$

Another representation of $d\varphi\wedge d\cos\theta$ is in terms of a 2-component complex unit vector $\boldsymbol{z} = (z_1, z_2)$ such that

$$\boldsymbol{S} = s\,\boldsymbol{z}^{*T}\,\boldsymbol{\sigma}\,\boldsymbol{z}$$

or $\quad S_i = s\,\boldsymbol{z}^{*T}\,\sigma_i\,\boldsymbol{z} \qquad$ with $\qquad |\,\boldsymbol{z}\,|^2 = \boldsymbol{z}^{*T}\,\boldsymbol{z} = z_1^*\,z_1 + z_2^*\,z_2 = z_\alpha^*\,z_\alpha = 1$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\to \quad S_1 = s\,(z_1^*, z_2^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = s\,(z_1^*, z_2^*)\begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = s\,(z_1^*\,z_2 + z_2^*\,z_1)$$

$$S_2 = s\,(z_1^*, z_2^*)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = s\,(z_1^*, z_2^*)\begin{pmatrix} -i\,z_2 \\ i\,z_1 \end{pmatrix} = i\,s\,(-z_1^*\,z_2 + z_2^*\,z_1)$$

$$S_3 = s\,(z_1^*, z_2^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = s\,(z_1^*, z_2^*)\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} = s\,(z_1^*\,z_1 - z_2^*\,z_2)$$

$$\therefore \quad \cos\theta = \frac{S_3}{s} = z_1^*\,z_1 - z_2^*\,z_2 \qquad \varphi = \tan^{-1}\frac{S_2}{S_1} = \tan^{-1}\left(i\,\frac{-z_1^*\,z_2 + z_2^*\,z_1}{z_1^*\,z_2 + z_2^*\,z_1}\right)$$

$$d\cos\theta = z_1^*\,dz_1 + z_1\,dz_1^* - z_2^*\,dz_2 - z_2\,dz_2^*$$

Using

$$1 - \left(\frac{-z_1^*\,z_2 + z_2^*\,z_1}{z_1^*\,z_2 + z_2^*\,z_1}\right)^2 = \frac{4\,z_1^*\,z_1\,z_2^*\,z_2}{(z_1^*\,z_2 + z_2^*\,z_1)^2}$$

$$d\left(\frac{-z_1^*\,z_2 + z_2^*\,z_1}{z_1^*\,z_2 + z_2^*\,z_1}\right) = (z_1^*\,z_2 + z_2^*\,z_1)^{-2}\,[\,(z_1^*\,z_2 + z_2^*\,z_1)\,(-z_1^*\,dz_2 - z_2\,dz_1^* + z_2^*\,dz_1 + z_1\,dz_2^*)$$

$$-\,(-z_1^*\,z_2 + z_2^*\,z_1)\,(z_1^*\,dz_2 + z_2\,dz_1^* + z_2^*\,dz_1 + z_1\,dz_2^*)\,]$$

$$= 2\,(z_1^*\,z_2 + z_2^*\,z_1)^{-2}\,[\,z_1^*\,z_2\,(z_2^*\,dz_1 + z_1\,dz_2^*) - z_2^*\,z_1\,(z_1^*\,dz_2 + z_2\,dz_1^*)\,]$$

$$= \frac{2\,z_1^*\,z_1\,z_2^*\,z_2}{(z_1^*\,z_2 + z_2^*\,z_1)^2}\left(\frac{dz_1}{z_1} + \frac{dz_2^*}{z_2^*} - \frac{dz_2}{z_2} - \frac{dz_1^*}{z_1^*}\right)$$

we have

$$d\varphi = \frac{i}{2}\left(\frac{dz_1}{z_1} + \frac{dz_2^*}{z_2^*} - \frac{dz_2}{z_2} - \frac{dz_1^*}{z_1^*}\right)$$

$$\to \quad d\varphi\wedge d\cos\theta = \frac{i}{2}\left(\frac{dz_1}{z_1} + \frac{dz_2^*}{z_2^*} - \frac{dz_2}{z_2} - \frac{dz_1^*}{z_1^*}\right)\wedge(z_1^*\,dz_1 + z_1\,dz_1^* - z_2^*\,dz_2 - z_2\,dz_2^*)$$

$$= \frac{i}{2}\left(d z_1 \wedge d z_1^* - \frac{z_2^*}{z_1} d z_1 \wedge d z_2 - \frac{z_2}{z_1} d z_1 \wedge d z_2^* \right.$$
$$+ \frac{z_1^*}{z_2^*} d z_2^* \wedge d z_1 + \frac{z_1}{z_2^*} d z_2^* \wedge d z_1^* - d z_2^* \wedge d z_2$$
$$- \frac{z_1^*}{z_2} d z_2 \wedge d z_1 - \frac{z_1}{z_2} d z_2 \wedge d z_1^* + d z_2 \wedge d z_2^*$$
$$\left. - d z_1^* \wedge d z_1 + \frac{z_2^*}{z_1^*} d z_1^* \wedge d z_2 + \frac{z_2}{z_1^*} d z_1^* \wedge d z_2^* \right)$$
$$= \frac{i}{2}\left[2 d z_1 \wedge d z_1^* + \left(\frac{z_1^*}{z_2} - \frac{z_2^*}{z_1} \right) d z_1 \wedge d z_2 - \left(\frac{z_1^*}{z_2^*} + \frac{z_2}{z_1} \right) d z_1 \wedge d z_2^* \right.$$
$$\left. + \left(\frac{z_2}{z_1^*} - \frac{z_1}{z_2^*} \right) d z_1^* \wedge d z_2^* + 2 d z_2 \wedge d z_2^* + \left(\frac{z_2^*}{z_1^*} + \frac{z_1}{z_2} \right) d z_1^* \wedge d z_2 \right]$$

(a)

$$z_1^* z_1 + z_2^* z_2 = 1$$
$$\rightarrow \quad z_1^* d z_1 + z_1 d z_1^* + z_2^* d z_2 + z_2 d z_2^* = 0$$

(b)

$d z_1 \wedge$ eq(b) gives

$$z_1 d z_1 \wedge d z_1^* + z_2^* d z_1 \wedge d z_2 + z_2 d z_1 \wedge d z_2^* = 0$$
$$- \frac{z_2^*}{z_1} d z_1 \wedge d z_2 - \frac{z_2}{z_1} d z_1 \wedge d z_2^* = d z_1 \wedge d z_1^*$$

$d z_1^* \wedge$ eq(b) gives

$$z_1^* d z_1^* \wedge d z_1 + z_2^* d z_1^* \wedge d z_2 + z_2 d z_1^* \wedge d z_2^* = 0$$
$$\frac{z_2^*}{z_1^*} d z_1^* \wedge d z_2 + \frac{z_2}{z_1^*} d z_1^* \wedge d z_2^* = d z_1 \wedge d z_1^*$$

$d z_2 \wedge$ eq(b) gives

$$z_1^* d z_2 \wedge d z_1 + z_1 d z_2 \wedge d z_1^* + z_2 d z_2 \wedge d z_2^* = 0$$
$$- \frac{z_1^*}{z_2} d z_2 \wedge d z_1 - \frac{z_1}{z_2} d z_2 \wedge d z_1^* = d z_2 \wedge d z_2^*$$

$d z_2^* \wedge$ eq(b) gives

$$z_1^* d z_2^* \wedge d z_1 + z_1 d z_2^* \wedge d z_1^* + z_2^* d z_2^* \wedge d z_2 = 0$$
$$\frac{z_1^*}{z_2^*} d z_2^* \wedge d z_1 + \frac{z_1}{z_2^*} d z_2^* \wedge d z_1^* = d z_2 \wedge d z_2^*$$

Eq(a) thus becomes

$$d \varphi \wedge d \cos\theta = 2 i (d z_1 \wedge d z_1^* + d z_2 \wedge d z_2^*)$$
$$= 2 i \, d z_\alpha \wedge d z_\alpha^*$$

It is straightforward to show that

$$\boldsymbol{z} = \left(e^{i \varphi/2} \cos\frac{\theta}{2}, e^{-i \varphi/2} \sin\frac{\theta}{2} \right)$$

Thus,

$$z_1^* z_1 - z_2^* z_2$$

A3.2.2. Quantization of Spin Degrees of Freedom

From eq(A3.9a), any path with $\cos\theta = \text{const}$ leaves $\mathcal{H}$ unchanged.

Consider the $\dot{\varphi}$ term in the action given by eq(A3.10)

$$\mathcal{I} = s\int dt\,\cos\theta\,\dot{\varphi} = s\int d\varphi\,\cos\theta$$

Hence, keeping $\cos\theta = \text{const}$, then

$$\Delta\mathcal{I} = s\cos\theta\,\Delta\varphi$$

However, of these paths, only the cases $\theta = 0$ & π correspond to a rotation about $\boldsymbol{S} = (\theta, \phi)$. In which case, if $\Delta\varphi = 2\pi$,

$$e^{i\mathcal{A}/\hbar} \to e^{i\mathcal{A}/\hbar}\,e^{\pm i2\pi s/\hbar} = \begin{cases} e^{i\mathcal{A}/\hbar} & \text{if } s/\hbar = \text{integer} \\ -e^{i\mathcal{A}/\hbar} & \text{if } s/\hbar = \text{half integer} \end{cases}$$

However, $e^{i\mathcal{A}/\hbar}$ should be unchanged since geometrically, everything is returned to its original state. This can be achieved while keeping the dynamics unchanged, if we add to eq(A3.10) a total derivative term so that

$$\mathcal{A} = \int dt\left(s\cos\theta\,\dot{\varphi} - \mathcal{H}\right) + \gamma\int d\varphi$$
$$= \int dt\left[(\gamma + s\cos\theta)\,\dot{\varphi} - \mathcal{H}\right]$$

(A3.11)

with

$$\gamma = \begin{cases} 0 & \text{if } s/\hbar = \text{integer} \\ \frac{1}{2}\hbar & \text{if } s/\hbar = \text{half integer} \end{cases}$$

To relate this action to the usual operator formulation of the angular momentum, we first note that, as classical vectors,

$$S_\pm = S_1 \pm i S_2$$
$$= s\sin\theta\,(\cos\varphi \pm i\sin\varphi)$$
$$= e^{\pm i\varphi}\,s\left(1 - \cos^2\theta\right)^{1/2}$$
$$= e^{\pm i\varphi}\left(s^2 - S_3^2\right)^{1/2}$$

(c)

Treated as an operator,

$$S_3 = p_\varphi = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

The quantization of $S_\pm$ runs into the problem of operator ordering. We shall first follow Zinn-Justin & choose

$$S_\pm = e^{\pm i\varphi/2}\left(s^2 - p_\varphi^2\right)^{1/2} e^{\pm i\varphi/2}$$

(d)

Note that the validity of our quantization scheme hinges on whether the operators satisfy the known commutation relations, e.g.,

$$[S_3, S_\pm] = [S_3, S_1 \pm i S_2] = i\hbar(S_2 \mp i S_1) = \pm\hbar(S_1 \pm i S_2) = \pm\hbar S_\pm \qquad \text{(e)}$$
$$[S_+, S_-] = [S_1, -i S_2] + [i S_2, S_1] = 2\hbar S_3$$

(f)

Thus,

$$\frac{1}{\hbar}p_\varphi S_\pm f = \left(\frac{p_\varphi}{\hbar}e^{\pm i\varphi/2}\right)\left(s^2 - p_\varphi^2\right)^{1/2} e^{\pm i\varphi/2} f + e^{\pm i\varphi/2}\left(s^2 - p_\varphi^2\right)^{1/2}\left(\frac{p_\varphi}{\hbar}e^{\pm i\varphi/2}\right) f$$
$$+ e^{\pm i\varphi/2}\left(s^2 - p_\varphi^2\right)^{1/2} e^{\pm i\varphi/2}\frac{p_\varphi}{\hbar} f\Big]$$

$$= \left(\pm \frac{1}{2} \pm \frac{1}{2} \right) S_\pm f + S_\pm \frac{p_\varphi}{\hbar} f$$

$$= \pm S_\pm f + S_\pm \frac{p_\varphi}{\hbar} f$$

→ $\quad [p_\varphi, S_\pm] f = \pm \hbar S_\pm f \qquad \forall f$

∴ $\quad [S_3, S_\pm] = \pm \hbar S_\pm$

so that eq(e) is satisfied.

From eq(d), we have

$$S_\pm S_\mp f = e^{\pm i \varphi/2} (s^2 - p_\varphi^2) e^{\mp i \varphi/2} f$$

Using

$$p_\varphi^2 e^{\pm i \varphi/2} f = p_\varphi e^{\pm i \varphi/2} \left(\pm \frac{\hbar}{2} + p_\varphi \right) f = e^{\pm i \varphi/2} \left(\pm \frac{\hbar}{2} + p_\varphi \right) \left(\pm \frac{\hbar}{2} + p_\varphi \right) f$$

$$= e^{\pm i \varphi/2} \left(\frac{\hbar^2}{4} \pm \hbar p_\varphi + p_\varphi^2 \right) f$$

we have

$$S_\pm S_\mp f = \left[s^2 - \left(\frac{\hbar^2}{4} \mp \hbar p_\varphi + p_\varphi^2 \right) \right] f$$

→ $\quad [S_+, S_-] = 2 \hbar p_\varphi = 2 \hbar S_3$

in agreement with eq(e). $S_\pm$ are therfore successfully quantized using eq(d).

Also,

$$S_+ S_- + S_- S_+ = 2 \left(s^2 - \frac{\hbar^2}{4} - p_\varphi^2 \right) = 2 \left(s^2 - \frac{\hbar^2}{4} - S_3^2 \right) \qquad \text{(g)}$$

On the other hand,

$$S_\pm S_\mp = (S_1 \pm i S_2)(S_1 \mp i S_2) = S_1^2 + S_2^2 \mp i [S_1, S_2] = S_1^2 + S_2^2 \pm \hbar S_3$$

→ $\quad S_1^2 + S_2^2 = \frac{1}{2} (S_+ S_- + S_- S_+) = S_+ S_- - \hbar S_3 = S_- S_+ + \hbar S_3$

$$\boldsymbol{S}^2 = S_1^2 + S_2^2 + S_3^2 = S_+ S_- - \hbar S_3 + S_3^2 = S_- S_+ + \hbar S_3 + S_3^2$$

$$= \frac{1}{2} (S_+ S_- + S_- S_+) + S_3^2$$

Using eq(g), we have

$$\boldsymbol{S}^2 = s^2 - \frac{\hbar^2}{4}$$

$$= j(j+1) \hbar^2 \qquad \text{(h)}$$

where $j(j+1)\hbar^2$ with $j = 0, \frac{1}{2}, 1, \ldots$ are the eigen-values of the quantum angular momentum operator $\boldsymbol{S}^2$.

Eq(h) can be written as

$$s^2 = \left(j^2 + j + \frac{1}{4} \right) \hbar^2$$

$$= \left(j + \frac{1}{2} \right)^2 \hbar^2$$

$$\to \qquad s = \left(j + \frac{1}{2}\right)\hbar$$

(A3.12)

i.e., after quantization, $\frac{s}{\hbar} = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

The eigenstates of S_3 are given by

$$S_3 \mid m\rangle = m\,\hbar \mid m\rangle$$

which, in the φ-representation, becomes

$$\frac{\hbar}{i}\frac{\partial}{\partial \varphi}\langle \varphi \mid m\rangle = m\,\hbar\,\langle \varphi \mid m\rangle$$

with

$$\langle \varphi \mid m\rangle = \frac{e^{i\,m\,\varphi}}{\sqrt{2\,\pi}}$$

The projector onto the subspace spanned by the orthonormal basis $\{\mid m\rangle\}$ belonging to the angular momentum j is

$$K = \sum_{m=-j}^{j} \mid m\rangle\langle m \mid$$

In the φ-representation,

$$K(\varphi'', \varphi') = \sum_{m=-j}^{j} \langle \varphi'' \mid m\rangle\langle m \mid \varphi'\rangle \tag{i}$$

$$= \frac{1}{2\,\pi}\sum_{m=-j}^{j} e^{i\,m\,(\varphi''-\varphi')}$$

$$= \frac{1}{2\,\pi}\,\frac{e^{-i\,j\,(\varphi''-\varphi')} - e^{i\,(j+1)\,(\varphi''-\varphi')}}{1 - e^{i\,(\varphi''-\varphi')}}$$

$$= \frac{1}{2\,\pi}\,\frac{e^{-i\,(j+1/2)\,(\varphi''-\varphi')} - e^{i\,(j+1/2)\,(\varphi''-\varphi')}}{e^{-i\,(\varphi''-\varphi')/2} - e^{i\,(\varphi''-\varphi')/2}}$$

$$= \frac{1}{2\,\pi}\,\frac{\sin\left[\left(j+\frac{1}{2}\right)(\varphi''-\varphi')\right]}{\sin\left[\frac{1}{2}(\varphi''-\varphi')\right]}$$

$$= \frac{1}{2\,\pi}\,\frac{\sin\left[\left(j+\frac{1}{2}\right)(\varphi'-\varphi'')\right]}{\sin\left[\frac{1}{2}(\varphi'-\varphi'')\right]}$$

(A3.13a)

$$= K(\varphi', \varphi'')$$

Note that eq(A3.13a) differs from Zinn-Justin's eq(A.13) by a normalization factor $\frac{1}{2\,\pi}$.

Orthonormal basis is necessary to ensure the idempotent property of a projection operator, namely,

$$K^2 = K$$

Eq(i) shows that $K(\varphi'', \varphi')$ is the sum of all probability amplitudes that go from φ' to φ'' via all possible states $\mid m\rangle$. It should be proportional to the path integral for the action in eq(A3.10) with no external field ($H=0$) and $p_\varphi = \text{const}$. Since the possible paths going φ' to φ'' differ from each other only in the number n of full turns they go around $\boldsymbol{S}$, we have

$$\mathcal{A} = \int d\,t\, p_\varphi\, \dot{\varphi} \;=\; p_\varphi\, \Delta\varphi \;= p_\varphi\,(\,\varphi'' - \varphi' + 2\,\pi\,n\,)$$

$$K(\varphi'',\, \varphi') \propto \sum_n \int_{-s}^{s} \frac{d\,p_\varphi}{\hbar}\, e^{i\,p_\varphi\,(\,\varphi''-\varphi'+2\,\pi\,n\,)/\hbar} \qquad\qquad (\; p_\varphi = s\cos\theta \in [-s,\, s]\;)$$

$$= \sum_n \frac{e^{i\,s\,(\,\varphi''-\varphi'+2\,\pi\,n\,)/\hbar} - e^{-i\,s\,(\,\varphi''-\varphi'+2\,\pi\,n\,)/\hbar}}{i\;(\,\varphi'' - \varphi' + 2\,\pi\,n\,)}$$

$$= \sum_n \frac{2\sin\left[\frac{s}{\hbar}\,(\,\varphi'' - \varphi' + 2\,\pi\,n\,)\right]}{\varphi'' - \varphi' + 2\,\pi\,n}$$

From eq(A3.12), we have

$$\sin\left[\,\frac{s}{\hbar}\,(\,\varphi'' - \varphi' + 2\,\pi\,n\,)\,\right] = \sin\left[\left(j+\frac{1}{2}\right)(\,\varphi'' - \varphi'\,) + \pi\,n\,(2\,j+1)\,\right]$$

$$= \sin\left[\left(j+\frac{1}{2}\right)(\,\varphi'' - \varphi'\,)\right]\cos[\pi\,n\,(2\,j+1)\,]$$

$$= \sin\left[\left(j+\frac{1}{2}\right)(\,\varphi'' - \varphi'\,)\right]\,(-)^n$$

$$\rightarrow \qquad K\,(\varphi'',\, \varphi') \propto 2\sin\left[\left(j+\frac{1}{2}\right)(\,\varphi'' - \varphi'\,)\right]\,\sum_n \frac{(-)^n}{\varphi'' - \varphi' + 2\,\pi\,n}$$

Using [see G-R, eq(1.422.3)]

$$\csc\pi\,x = \frac{1}{\pi\,x} + \frac{2\,x}{\pi}\sum_{k=1}^{\infty} \frac{(-)^k}{x^2 - k^2}$$

$$= \frac{1}{\pi\,x} + \frac{1}{\pi}\sum_{k=1}^{\infty} (-)^k \left(\frac{1}{x-k} + \frac{1}{x+k}\right)$$

$$= \frac{1}{\pi}\sum_{k=-\infty}^{\infty} \frac{(-)^k}{x+k}$$

we have

$$\sum_n \frac{(-)^n}{\varphi'' - \varphi' + 2\,\pi\,n} = \frac{1}{2\,\pi}\sum_n \frac{(-)^n}{\frac{\varphi''-\varphi'}{2\,\pi} + n} = \frac{1}{2}\csc[\,(\varphi'' - \varphi')/2\,]$$

$$K\,(\varphi'',\, \varphi') \propto \frac{\sin\left[\left(j+\frac{1}{2}\right)(\,\varphi'' - \varphi'\,)\right]}{\sin[\,(\varphi'' - \varphi')/2\,]}$$

which agrees with eq(A3.13a), as declared.