A3.3. The Magnetic Monopole

The interaction between a particle & a magnetic field $\boldsymbol{B}$ is given by the Lagrangian

$$\mathcal{L}_{\text{mag}} = e\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A} \qquad \text{with} \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}$$

& action

$$\mathcal{A}_{\text{mag}} = e\int dt\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A} \;= e\int d\boldsymbol{x}\cdot\boldsymbol{A}$$

$$= e\int A_i\,dx_i \qquad \text{(a)}$$

The space part of the field tensor is defined as

$$F_{ij} = \partial_i A_j - \partial_j A_i \qquad (\,i,\ j = 1,\ 2,\ 3\,)$$

$$= \varepsilon_{ijk}\,B_k$$

Proof:

$$\varepsilon_{ijk}\,B_k = \varepsilon_{ijk}\,\varepsilon_{kmn}\,\partial_m A_n$$

$$= (\,\delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm}\,)\,\partial_m A_n$$

$$= \partial_i A_j - \partial_j A_i \qquad \text{QED}$$

From the 1-form

$$\omega = A_i\,dx_i$$

we get the 2-form

$$d\omega = \partial_j A_i\,dx_j\wedge dx_i$$

$$= \frac{1}{2}\left(\partial_i A_j - \partial_j A_i\right)dx_i\wedge dx_j$$

Hence,

$$\frac{1}{2}\,e\int F_{ij}\,dx_i\wedge dx_j = \frac{1}{2}\,e\int\left(\partial_i A_j - \partial_j A_i\right)dx_i\wedge dx_j$$

$$= e\int\omega \;= e\int A_i\,dx_i$$

$$= \mathcal{A}_{\text{mag}} \qquad \text{(b)}$$

$$= \frac{1}{2}\,e\int\varepsilon_{ijk}\,B_k\,dx_i\wedge dx_j$$

$$= e\int\boldsymbol{B}\cdot d\boldsymbol{\sigma} \qquad \text{(c)}$$

where $d\boldsymbol{\sigma}$ is a surface element.

The above is explicitly derived for a Euclidean space described by Cartesian coordinates. Eq(a) is therefore always valid locally while eq(b), with proper placement of contra- & co-variant components, is applicable everywhere in any curved space of arbitrary topology.

Analogous to the electric case, a magnetic charge (or monopole) g generates a magnetic field

$$\boldsymbol{B} = g\,\frac{\boldsymbol{x}}{|\,\boldsymbol{x}\,|^3} \qquad \text{(d)}$$

which satisfies the modified Maxwell eqs

$$\nabla\cdot\boldsymbol{B} = 4\,\pi\,g\,\delta(\boldsymbol{x}) \qquad \text{(e)}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = 0$$

& $\qquad \nabla\times\boldsymbol{B} = 0$

In spherical coordinates, eq(d) becomes

$$\boldsymbol{B} = \frac{g}{r^2}\,\hat{\boldsymbol{r}}$$

If we integrate over a spherical surface of radius r centered at the monopole,

$$\frac{1}{2}\int F_{ij}\, d\, x_i \wedge d\, x_j = \int_0^{2\pi} d\,\varphi \int_{-1}^{1} d\cos\theta\, r^2 \frac{g}{r^2}$$

$$= 4\,\pi\, g \qquad \text{(f)}$$

which is simply the Gauss law for eqs(d & e).

Since

$$\nabla \cdot \boldsymbol{B} = \nabla \cdot (\nabla \times \boldsymbol{A}) = 0$$

eq(e or f) implies that $\boldsymbol{A}$ cannot be defined at the monopole.

A family of solutions of $\boldsymbol{A}$ that gives rise to eq(d) is

$$A_i(\boldsymbol{x}) = g\, \varepsilon_{ijk}\, n_k\, x_j \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]} \qquad \text{(g)}$$

where $\boldsymbol{n}$ is any constant unit vector.

Using

$$\partial_j r = \frac{x_j}{r}$$

$$\partial_j \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]} = \frac{n_j}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]} - \frac{x_j\, \boldsymbol{n}\cdot\boldsymbol{x}}{r^3\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]} - \frac{2\,(\boldsymbol{n}\cdot\boldsymbol{x})\left[x_j - n_j(\boldsymbol{n}\cdot\boldsymbol{x})\right]}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]^2}$$

$$= \left(n_j\, r^4 + r^2\, n_j(\boldsymbol{n}\cdot\boldsymbol{x})^2 - 3\, r^2\, x_j\, (\boldsymbol{n}\cdot\boldsymbol{x}) + x_j(\boldsymbol{n}\cdot\boldsymbol{x})^3\right) \Big/ r^3 \left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]^2$$

we have

$$B_i = g\, \varepsilon_{ijk}\, \partial_j\, \varepsilon_{kmn}\, n_n\, x_m \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]}$$

$$= g\, \varepsilon_{ijk}\, \varepsilon_{kmn}\, n_n \left(\delta_{jm} \frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r\left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]} + x_m \left(\left(n_j\, r^4 + r^2\, n_j(\boldsymbol{n}\cdot\boldsymbol{x})^2 - 3\, r^2\, x_j\, (\boldsymbol{n}\cdot\boldsymbol{x}) + x_j(\boldsymbol{n}\cdot\boldsymbol{x})^3\right) \Big/ r^3 \left[r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2\right]^2\right)\right)$$

Let

$$\boldsymbol{A} = f\, \boldsymbol{c} \qquad (1)$$

Then

$$\nabla \times \boldsymbol{A} = \nabla f \times \boldsymbol{c} + f\, \nabla \times \boldsymbol{c} = \frac{g}{r^2}\hat{\boldsymbol{r}} \qquad (2)$$

$$\rightarrow \qquad \boldsymbol{c}\cdot(\nabla f \times \boldsymbol{c}) + f\boldsymbol{c}\cdot(\nabla \times \boldsymbol{c}) = 0 = \frac{g}{r^2}\, \boldsymbol{c}\cdot\hat{\boldsymbol{r}}$$

which can be satisfied if we set

$$\boldsymbol{c} = \hat{\boldsymbol{n}} \times \boldsymbol{r} \qquad (3)$$

where $\hat{\boldsymbol{n}}$ is any constant unit vector.

Eq(2) also implies

$$\boldsymbol{r}\cdot(\nabla f \times \boldsymbol{c}) + f\boldsymbol{r}\cdot(\nabla \times \boldsymbol{c}) = \frac{g}{r} \qquad (4)$$

Using

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}$$

we have

$$\nabla f \times \boldsymbol{c} = \nabla f \times \left(\hat{\boldsymbol{n}} \times \boldsymbol{r}\right) = (\boldsymbol{r}\cdot\nabla f)\,\hat{\boldsymbol{n}} - \left(\hat{\boldsymbol{n}}\cdot\nabla f\right)\boldsymbol{r} \qquad (5)$$

$$\boldsymbol{r}\cdot(\nabla f \times \boldsymbol{c}) = (\boldsymbol{r}\cdot\nabla f)\left(\boldsymbol{r}\cdot\hat{\boldsymbol{n}}\right) - \left(\hat{\boldsymbol{n}}\cdot\nabla f\right) r^2$$

Using

$$\nabla\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{a}\,(\nabla\cdot\boldsymbol{b})-\boldsymbol{b}\,(\nabla\cdot\boldsymbol{a})+(\boldsymbol{b}\cdot\nabla)\,\boldsymbol{a}-(\boldsymbol{a}\cdot\nabla)\,\boldsymbol{b}$$

we have

$$\nabla\times\boldsymbol{c}=\nabla\times\left(\hat{\boldsymbol{n}}\times\boldsymbol{r}\right)\;=\hat{\boldsymbol{n}}\,(\nabla\cdot\boldsymbol{r})-\left(\hat{\boldsymbol{n}}\cdot\nabla\right)\boldsymbol{r}\;=3\,\hat{\boldsymbol{n}}\;-\hat{\boldsymbol{n}}\;=2\,\hat{\boldsymbol{n}} \qquad (6)$$

Eq(4) thus becomes

$$(\boldsymbol{r}\cdot\nabla f)\left(\boldsymbol{r}\cdot\hat{\boldsymbol{n}}\right)-\left(\hat{\boldsymbol{n}}\cdot\nabla f\right)r^2+2f\boldsymbol{r}\cdot\hat{\boldsymbol{n}}=\frac{g}{r} \qquad (7)$$

Putting eqs(3, 5 & 6) into eq(2) gives

$$\nabla\times\boldsymbol{A}=(\boldsymbol{r}\cdot\nabla f)\,\hat{\boldsymbol{n}}-\left(\hat{\boldsymbol{n}}\cdot\nabla f\right)\boldsymbol{r}+2\,f\,\hat{\boldsymbol{n}}=\frac{g}{r^2}\,\hat{\boldsymbol{r}} \qquad (8)$$

Eq(8) implies

$$(\boldsymbol{r}\cdot\nabla f)\left(\boldsymbol{r}\times\hat{\boldsymbol{n}}\right)+2\,f\left(\boldsymbol{r}\times\hat{\boldsymbol{n}}\right)=\frac{g}{r^2}\,\hat{\boldsymbol{r}}\times\hat{\boldsymbol{r}}=0$$

Allowing ***n*** to be arbitrary, we have

$$(\boldsymbol{r}\cdot\nabla f)+2\,f=0 \qquad (9)$$

Eq(7) thus becomes

$$\hat{\boldsymbol{n}}\cdot\nabla f=-\frac{g}{r^3} \qquad (10)$$

Let

$$\hat{\boldsymbol{n}}=(\,\sin\theta_0\cos\phi_0\,,\;\sin\theta_0\sin\phi_0,\;\cos\theta_0\,)$$

$$X=\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}=\cos\eta$$

Using

$$\hat{\boldsymbol{r}}=(\,\sin\theta\cos\phi\,,\;\sin\theta\sin\phi,\;\cos\theta\,)$$

$$\hat{\boldsymbol{\theta}}=(\,\cos\theta\cos\phi\,,\;\cos\theta\sin\phi,\;-\sin\theta\,)$$

$$\hat{\boldsymbol{\phi}}=(\,-\sin\phi\,,\;\cos\phi,\;0\,)$$

we have

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}&=\sin\theta_0\sin\theta\,(\,\cos\phi_0\cos\phi+\sin\phi_0\sin\phi)+\cos\theta_0\cos\theta\\&=\sin\theta_0\sin\theta\cos\left(\phi_0-\phi\right)+\cos\theta_0\cos\theta\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\theta}}&=\sin\theta_0\cos\phi_0\cos\theta\cos\phi+\sin\theta_0\sin\phi_0\cos\theta\sin\phi-\cos\theta_0\sin\theta\\&=\sin\theta_0\cos\theta\cos\left(\phi_0-\phi\right)-\cos\theta_0\sin\theta\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\phi}}&=-\sin\theta_0\cos\phi_0\sin\phi+\sin\theta_0\sin\phi_0\cos\phi\\&=\sin\theta_0\sin(\phi_0-\phi)\end{aligned}$$

$$\therefore\qquad \frac{\partial\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}}{\partial\,r}=0$$

$$\frac{\partial\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}}{\partial\,\theta}=\sin\theta_0\cos\theta\cos(\phi_0-\phi)-\cos\theta_0\sin\theta\;=\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\theta}}$$

$$\frac{1}{\sin\theta}\,\frac{\partial\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}}{\partial\,\phi}=\sin\theta_0\sin(\phi_0-\phi)=\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\phi}}$$

Let

$$f=p\left(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}\right)h(r)=p(X)\,h(r) \qquad (11)$$

then

$$\nabla f=p\,\frac{d\,h}{d\,r}\,\hat{\boldsymbol{r}}+\frac{1}{r}\,\frac{\partial\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}}{\partial\,\theta}\,\frac{d\,p}{d\,X}\,h\,\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\,\frac{\partial\,\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{r}}}{\partial\,\phi}\,\frac{d\,p}{d\,X}\,h\,\hat{\boldsymbol{\phi}}$$

$$= p \frac{dh}{dr} \hat{\boldsymbol{r}} + \frac{dp}{dX} \frac{h}{r} \left[\left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\theta}} \right) \hat{\boldsymbol{\theta}} + \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\phi}} \right) \hat{\boldsymbol{\phi}} \right]$$

$$\to \qquad \hat{\boldsymbol{n}} \cdot \nabla f = p \frac{dh}{dr} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} + \frac{dp}{dX} \frac{h}{r} \left[\left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\theta}} \right)^2 + \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\phi}} \right)^2 \right]$$

Since $\left\{ \hat{\boldsymbol{r}}, \ \hat{\boldsymbol{\theta}}, \ \hat{\boldsymbol{\phi}} \right\}$ is an orthnormal basis,

$$\left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} \right)^2 + \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\theta}} \right)^2 + \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\phi}} \right)^2 = \hat{\boldsymbol{n}}^2 = 1$$

Eq(10) becomes

$$\hat{\boldsymbol{n}} \cdot \nabla f = p \frac{dh}{dr} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} + \frac{dp}{dX} \frac{h}{r} \left[1 - \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} \right)^2 \right]$$

$$= p \frac{dh}{dr} X + \frac{dp}{dX} \frac{h}{r} \left(1 - X^2 \right)$$

$$= -\frac{g}{r^3} \qquad (12)$$

The dependence on r can be eliminated if we set

$$h = \frac{1}{r^2} \qquad (13)$$

so that eq(11) becomes

$$-2 p X + \frac{dp}{dX} \left(1 - X^2 \right) = -\frac{g}{n} \qquad (14)$$

The general solution of eq(14) can be obtained using the *Mathematica* code

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DSolve[ -2 X p[X] + p'[X] (1 - X^2) - g/n == 0, p, X]
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which gives

$$p = \frac{g X}{n \left(1 - X^2 \right)} + \frac{\alpha}{1 - X^2}$$

where α is an arbitrary constant.

To simplify thing, we'll use only the particular solution (obtained by setting $\alpha = 0$). Together with eqs(3,11 & 13), eq(1) becomes

$$\boldsymbol{A} = \hat{\boldsymbol{n}} \times \boldsymbol{r} \frac{g X}{r^2 \left(1 - X^2 \right)}$$

$$= g \hat{\boldsymbol{n}} \times \boldsymbol{r} \frac{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}}}{r^2 \left[1 - \left(\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} \right)^2 \right]}$$

$$= g \hat{\boldsymbol{n}} \times \boldsymbol{r} \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{r}}{r^2 \left[\boldsymbol{r}^2 - \left(\hat{\boldsymbol{n}} \cdot \boldsymbol{r} \right)^2 \right]} \qquad (15)$$

Note that $\boldsymbol{A}$ is singular if

$$\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{r}} = 1$$

i.e., for any $\boldsymbol{r}$ on a line that goes through the origin in the direction of $\hat{\boldsymbol{n}}$. Since $\hat{\boldsymbol{n}}$ is arbitrary, this singularity has no physical significance.

Using eq(15), the action eq(a) becomes

$$\mathcal{A}_{\text{mag}} = g\, e \int d\boldsymbol{x} \cdot (\hat{\boldsymbol{n}} \times \boldsymbol{r}) \frac{\hat{\boldsymbol{n}} \cdot \boldsymbol{r}}{r^2 \left[\boldsymbol{r}^2 - (\hat{\boldsymbol{n}} \cdot \boldsymbol{r})^2 \right]}$$

Since the integral is defined only $\text{mod}\, 4\,\pi$, the path integral weight factor $e^{i\,\mathcal{A}/\hbar}$ is well-defined only if

$4\,\pi\, g\, e\,/\hbar = 0 \quad \text{mod}\, 2\,\pi$

→ $2\, g\, e/\hbar = \text{integer}$

which is simply Dirac’s quantization condition.