n-Sphere

Ref: T.Frankel, “The Geometry of Physics”, Cambridge U Press (2004).

Spherical Angles

For an $(N-1)$-D sphere S^{N-1} embedded in N-D space, the spherical cordinates are given by

$\{\theta_1, \ .., \ \theta_{N-2}, \phi\}$ where $\theta_1, \ ..., \ \theta_{N-2} \in [0, \pi]$ but $\phi \in [0, 2\pi]$

Given N Cartesian axes, the spherical angles of a unit vector $\hat{\boldsymbol{r}}$ are defined as follows.

1. θ_1 is defined as the angle between $\hat{\boldsymbol{r}}$ & $\hat{\boldsymbol{x}}_1$, i.e., $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_1 = \cos\theta_1$

   $\rightarrow \quad x_1 = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_1 = \cos\theta_1$

2. Let $\boldsymbol{\rho}_1$ be the projection of $\hat{\boldsymbol{r}}$ into the subspace orthogonal to $\hat{\boldsymbol{x}}_1$.

   $\rightarrow \quad \hat{\boldsymbol{r}} = \rho_1 \hat{\boldsymbol{\rho}}_1 + x_1 \hat{\boldsymbol{x}}_1$

   $\therefore \quad 1 = \rho_1 \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\rho}}_1 + x_1 \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_1 = \rho_1^2 + \cos^2\theta_1$

   $\rho_1 = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\rho}}_1 = \sin\theta_1$ & $\boldsymbol{\rho}_1 = \sin\theta_1 \hat{\boldsymbol{\rho}}_1$

   θ_2 is defined to be the angle between $\boldsymbol{\rho}_1$ & $\hat{\boldsymbol{x}}_2$, i.e.,

   $\hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{x}}_1 = \sin\theta_1$ $\hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{x}}_2 = \cos\theta_2$

   $\rightarrow \quad x_2 = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_2 = \left(\rho_1 \hat{\boldsymbol{\rho}}_1 + x_1 \hat{\boldsymbol{x}}_1\right) \cdot \hat{\boldsymbol{x}}_2 = \rho_1 \hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{x}}_2 = \sin\theta_1 \cos\theta_2$

3. Let $\boldsymbol{\rho}_{12}$ be the projection of $\boldsymbol{\rho}_1$ into the subspace orthogonal to $\hat{\boldsymbol{x}}_2$ & hence to $\{\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2\}$.

   $\rightarrow \quad \hat{\boldsymbol{\rho}}_1 = \rho_{12} \hat{\boldsymbol{\rho}}_{12} + x_2 \hat{\boldsymbol{x}}_2$

   $1 = \rho_{12} \hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{\rho}}_{12} + x_2 \hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{x}}_2 = \rho_{12}^2 + \cos^2\theta_2$

   $\therefore \quad \rho_{12} = \hat{\boldsymbol{\rho}}_1 \cdot \hat{\boldsymbol{\rho}}_{12} = \sin\theta_2$ & $\boldsymbol{\rho}_{12} = \rho_1 \rho_{12} \hat{\boldsymbol{\rho}}_{12} = \sin\theta_1 \sin\theta_2 \hat{\boldsymbol{\rho}}_{12}$

   θ_3 is defined to be the angle between $\boldsymbol{\rho}_{12}$ & $\hat{\boldsymbol{x}}_3$, i.e.,

   $\hat{\boldsymbol{\rho}}_{12} \cdot \hat{\boldsymbol{x}}_2 = \sin\theta_2$ $\hat{\boldsymbol{\rho}}_{12} \cdot \hat{\boldsymbol{x}}_3 = \cos\theta_3$

   $\rightarrow \quad x_3 = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_3 = \left[\rho_1\left(\rho_{12} \hat{\boldsymbol{\rho}}_{12} + x_2 \hat{\boldsymbol{x}}_2\right) + x_1 \hat{\boldsymbol{x}}_1\right] \cdot \hat{\boldsymbol{x}}_3 = \rho_1 \rho_{12} \hat{\boldsymbol{\rho}}_{12} \cdot \hat{\boldsymbol{x}}_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3$

4. For $j = 2, \ ..., \ N-2$, the above scheme applies with $\hat{\boldsymbol{\rho}}_0 = \hat{\boldsymbol{r}}$.

   Let $\boldsymbol{\rho}_{1...j-1}$ be the projection of $\boldsymbol{\rho}_{1...j-2}$ into the subspace orthogonal to $\hat{\boldsymbol{x}}_{j-1}$ & hence to $\{\hat{\boldsymbol{x}}_1, \ ..., \hat{\boldsymbol{x}}_{j-1}\}$.

   $\rightarrow \quad \hat{\boldsymbol{\rho}}_{1...j-2} = \rho_{1...j-1} \hat{\boldsymbol{\rho}}_{1...j-1} + x_{j-1} \hat{\boldsymbol{x}}_{j-1}$

   $\therefore \quad 1 = \rho_{1...j-1} \hat{\boldsymbol{\rho}}_{1...j-2} \cdot \hat{\boldsymbol{\rho}}_{1...j-1} + x_{j-1} \hat{\boldsymbol{\rho}}_{1...j-2} \cdot \hat{\boldsymbol{x}}_{j-1} = \rho_{1...j-1}^2 + \cos^2\theta_{j-1}$

   $\rho_{1...j-1} = \hat{\boldsymbol{\rho}}_{1...j-2} \cdot \hat{\boldsymbol{\rho}}_{1...j-1} = \sin\theta_{j-1}$ & $\boldsymbol{\rho}_{1...j-1} = \sin\theta_{j-1} \hat{\boldsymbol{\rho}}_{1...j-1}$

   θ_j is defined to be the angle between $\hat{\boldsymbol{\rho}}_{1...j-1}$ & $\hat{\boldsymbol{x}}_j$, i.e.,

   $\hat{\boldsymbol{\rho}}_{1...j-1} \cdot \hat{\boldsymbol{x}}_{j-1} = \sin\theta_{j-1}$ $\hat{\boldsymbol{\rho}}_{1...j-1} \cdot \hat{\boldsymbol{x}}_j = \cos\theta_j$

   $\rightarrow \quad x_j = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_j = \rho_1 \rho_{12} \ldots \rho_{1...j-1} \hat{\boldsymbol{\rho}}_{1...j-1} \cdot \hat{\boldsymbol{x}}_j = \sin\theta_1 \ldots \sin\theta_{j-1} \cos\theta_j$

5. Finally, for $j = N-1$.

   Let $\hat{\boldsymbol{\rho}}_{1...N-2}$ be the projection of $\hat{\boldsymbol{\rho}}_{1...N-3}$ into the subspace orthogonal to $\hat{\boldsymbol{x}}_{N-2}$ & hence to $\{\hat{\boldsymbol{x}}_1, \ ..., \hat{\boldsymbol{x}}_{N-2}\}$.

   $\rightarrow \quad \hat{\boldsymbol{\rho}}_{1...N-3} = \rho_{1...N-2} \hat{\boldsymbol{\rho}}_{1...N-2} + x_{N-2} \hat{\boldsymbol{x}}_{N-2}$

   $\therefore \quad 1 = \rho_{1...N-2} \hat{\boldsymbol{\rho}}_{1...N-3} \cdot \hat{\boldsymbol{\rho}}_{1...n-2} + x_{N-2} \hat{\boldsymbol{\rho}}_{1...N-3} \cdot \hat{\boldsymbol{x}}_{N-2} = \rho_{1...N-2}^2 + \cos^2\theta_{N-2}$

   $\rho_{1...N-2} = \hat{\boldsymbol{\rho}}_{1...N-3} \cdot \hat{\boldsymbol{\rho}}_{1...N-2} = \sin\theta_{N-2}$ & $\boldsymbol{\rho}_{1...N-2} = \sin\theta_{N-2} \hat{\boldsymbol{\rho}}_{1...N-2}$

Up to now, the ranges of the angles $\theta_1, \ldots, \theta_{N-2}$ are all from 0 to π.

However, $\hat{\boldsymbol{\rho}}_{1\ldots N-2}$ is in the last plane $\{\hat{\boldsymbol{x}}_{N-1}, \hat{\boldsymbol{x}}_N\}$ and its components are more naturally described by the azimuthal angle $\phi \in [0, 2\pi]$. Hence, we define ϕ to be the angle between $\hat{\boldsymbol{\rho}}_{1\ldots N-2}$ & $\hat{\boldsymbol{x}}_{N-1}$, so that

$$\hat{\boldsymbol{\rho}}_{1\ldots N-2} \cdot \hat{\boldsymbol{x}}_{N-1} = \cos\phi \qquad \hat{\boldsymbol{\rho}}_{1\ldots N-2} \cdot \hat{\boldsymbol{x}}_N = \sin\phi$$

With

$$\hat{\boldsymbol{\rho}}_{1\ldots N-2} \cdot \hat{\boldsymbol{x}}_{N-2} = \sin\theta_{N-2}$$

we have

$$x_{N-1} = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_{N-1} = \rho_1\,\rho_{12}\ldots\rho_{1\ldots N-2}\,\hat{\boldsymbol{\rho}}_{1\ldots N-2} \cdot \hat{\boldsymbol{x}}_{N-1} = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-2}\cos\phi$$

$$x_N = \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{x}}_N = \rho_1\,\rho_{12}\ldots\rho_{1\ldots N-2}\,\hat{\boldsymbol{\rho}}_{1\ldots N-2} \cdot \hat{\boldsymbol{x}}_N = \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{N-2}\sin\phi$$

To summarize, the Cartesian coordinates of an N-D point can be expressed in terms of the spherical coordinates as

$$\begin{aligned}
x_1 &= r\cos\theta_1 \\
x_2 &= r\sin\theta_1\cos\theta_2 \\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\
&\vdots \\
x_j &= r\sin\theta_1\ldots\sin\theta_{j-1}\cos\theta_j = r\cos\theta_j\prod_{k=1}^{j-1}\sin\theta_k \\
&\vdots \\
x_{N-2} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-3}\cos\theta_{N-2} \\
x_{N-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2}\cos\phi \\
x_N &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2}\sin\phi
\end{aligned} \tag{1}$$

Note that

$$\begin{aligned}
x_{N-1}^2 + x_N^2 &= (r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2})^2 \\
&= r^2\prod_{k=1}^{N-2}\sin^2\theta_k
\end{aligned} \tag{2}$$

is independent of ϕ.

Also, for $N = 3$, our convention differs from the usual coordinate assignments:

$$x_1 = z \qquad x_2 = x \qquad x_3 = y$$

This difference is immaterial since both are right-handed.

Theorem I

For $1 \le m \le N-2$,

$$\sum_{j=m}^{N} x_j^2 = r^2\prod_{k=1}^{m-1}\sin^2\theta_k \tag{T1}$$

From eq(1), we have, for $1 \le j \le N-2$,

$$x_j^2 = r^2\cos^2\theta_j\prod_{k=1}^{j-1}\sin^2\theta_k = r^2\left(-\prod_{k=1}^{j}\sin^2\theta_k + \prod_{k=1}^{j-1}\sin^2\theta_k\right)$$

Together with eq(2), we have, for $1 \le m \le N-2$,

$$\sum_{j=m}^{N} x_j^2 = r^2 \left(\prod_{k=1}^{N-2} \sin^2\theta_k + \sum_{j=m}^{N-2} x_j^2 \right)$$
$$= r^2 \left(\prod_{k=1}^{N-2} \sin^2\theta_k - \prod_{k=1}^{N-2} \sin^2\theta_k + \prod_{k=1}^{N-3} \sin^2\theta_k - \ldots - \prod_{k=1}^{m} \sin^2\theta_k + \prod_{k=1}^{m-1} \sin^2\theta_k \right)$$
$$= r^2 \prod_{k=1}^{m-1} \sin^2\theta_k \qquad \text{QED}$$

Setting $m = 1$, we have

$$\sum_{j=1}^{N} x_j^2 = r^2$$

as expected.

Metric Tensor for S^{N-1} & Surface Element

The metric tensor on the sphere is given by (see, Frankel §2.8f)

$$g_{ij} = \sum_{k=1}^{N} \frac{\partial x_k}{\partial \theta_i} \frac{\partial x_k}{\partial \theta_j} \qquad i, j = 1, \ldots, N-1 \qquad \text{with } \theta_{N-1} = \phi$$
$$= g_{ji}$$

Using eq(1), we have

For $i, k = 1, \ldots, N-2$,

$$\frac{\partial x_k}{\partial \theta_i} = \begin{cases} x_k \cot\theta_i & \text{for } i < k \\ -x_k \tan\theta_k & \text{for } i = k \\ 0 & \text{for } i > k \end{cases} \qquad (3)$$

$$\frac{\partial x_{N-1}}{\partial \theta_i} = x_{N-1} \cot\theta_i \qquad \frac{\partial x_N}{\partial \theta_i} = x_N \cot\theta_i$$
$$\frac{\partial x_k}{\partial \phi} = 0 \qquad \frac{\partial x_{N-1}}{\partial \phi} = -x_{N-1} \tan\phi$$
$$\frac{\partial x_N}{\partial \phi} = x_N \cot\phi$$

From eq(3), we have, for $N-2 \geq j > i$,

$$\sum_{k=1}^{N-2} \frac{\partial x_k}{\partial \theta_i} \frac{\partial x_k}{\partial \theta_j} = \sum_{k \geq j}^{N-2} \frac{\partial x_k}{\partial \theta_i} \frac{\partial x_k}{\partial \theta_j} = -x_j^2 \cot\theta_i \tan\theta_j + \sum_{k>j}^{N-2} x_k^2 \cot\theta_i \cot\theta_j$$

$$\rightarrow \qquad g_{ij} = \left(x_{N-1}^2 + x_N^2\right) \cot\theta_i \cot\theta_j - x_j^2 \cot\theta_i \tan\theta_j + \sum_{k>j}^{N-2} x_k^2 \cot\theta_i \cot\theta_j$$
$$= -x_j^2 \cot\theta_i \tan\theta_j + \cot\theta_i \cot\theta_j \sum_{k>j}^{N} x_k^2$$
$$= \cot\theta_i \cot\theta_j \left(-x_j^2 \tan^2\theta_j + \sum_{k=j+1}^{N} x_k^2 \right)$$
$$= r^2 \cot\theta_i \cot\theta_j \left(-\prod_{k=1}^{j} \sin^2\theta_k + \prod_{k=1}^{j} \sin^2\theta_k \right) \qquad \text{[Eqs(1 \& T1) used]}$$

$= 0$

For $i \le N-2$,

$$g_{i,N-1} = \frac{\partial x_{N-1}}{\partial \theta_i}\frac{\partial x_{N-1}}{\partial \phi} + \frac{\partial x_N}{\partial \theta_i}\frac{\partial x_N}{\partial \phi}$$
$$= -x_{N-1}^2 \cot\theta_i \tan\phi + x_N^2 \cot\theta_i \cot\phi$$
$$= \left(-x_{N-1}^2 \tan^2\phi + x_N^2\right)\cot\theta_i \cot\phi$$
$$= \left(-\cos^2\phi \tan^2\phi + \sin^2\phi\right)\left(r^2 \prod_{k=1}^{N-2} \sin^2\theta_k\right)\cot\theta_i \cot\phi$$
$$= 0$$

Hence, the symmetric metric tensor $\boldsymbol{g}$ is diagonal.

From eq(3), we have, for $1 \le i \le N-2$,

$$g_{ii} = \sum_{k=1}^{N}\left(\frac{\partial x_k}{\partial \theta_i}\right)^2 = x_i^2 \tan^2\theta_i + \sum_{k=i+1}^{N} x_k^2 \cot^2\theta_i$$
$$= r^2 \tan^2\theta_i \cos^2\theta_i \prod_{k=1}^{i-1}\sin^2\theta_k + r^2\cot^2\theta_i \prod_{k=1}^{i}\sin^2\theta_k \qquad \text{[Eqs(1 \& T1) used]}$$
$$= r^2\left(1 + \cot^2\theta_i\right)\prod_{k=1}^{i}\sin^2\theta_k$$
$$= r^2 \prod_{k=1}^{i-1}\sin^2\theta_k$$

(4a)

$$g_{N-1,N-1} = \sum_{k=1}^{N}\left(\frac{\partial x_k}{\partial \phi}\right)^2 = x_{N-1}^2\tan^2\phi + x_N^2\cot^2\phi$$
$$= \left(\cos^2\phi \tan^2\phi + \sin^2\phi \cot^2\phi\right) r^2 \prod_{k=1}^{N-2}\sin^2\theta_k$$
$$= r^2 \prod_{k=1}^{N-2}\sin^2\theta_k$$

(4b)

Hence, g_{ij} can be treated as an $(N-1)\times(N-1)$ matrix $\boldsymbol{g}$.

$$\boldsymbol{g} = g_{ij} = r^2\,\mathrm{diag}\left(1,\ \sin^2\theta_1,\ \sin^2\theta_1\sin^2\theta_2,\ \ldots,\ \ \sin^2\theta_1\ldots\sin^2\theta_{N-2}\right)$$

(4)

$$\boldsymbol{g}^{-1} = g^{ij} = \frac{1}{r^2}\,\mathrm{diag}\left(1,\ \frac{1}{\sin^2\theta_1},\ \frac{1}{\sin^2\theta_1\sin^2\theta_2},\ \ldots,\ \ \frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{N-2}}\right)$$

$$g = \det\boldsymbol{g} = r^{2(N-1)}\prod_{j=2}^{N-1}\left(\prod_{k=1}^{j-1}\sin^2\theta_k\right)$$

Since θ_m appears once for each $j > m$, the power of $\sin^2\theta_m$ in g is $\sum_{j=m+1}^{N-1} 1 = N-m-1$.

$$\therefore \qquad g = r^{2(N-1)} \prod_{j=1}^{N-2} \sin^{2(N-j-1)} \theta_j$$

(4c)

$$= \left(r \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \ldots \sin\theta_{N-2} \right)^2$$

The surface element is therefore

$$d\,S_{N-1} = \sqrt{g}\ d\,\theta_1\, d\,\theta_2 \ldots d\,\theta_{N-2}\, d\,\phi$$

$$= r^{N-1} \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \ldots \sin\theta_{N-2}\ d\,\theta_1\, d\,\theta_2 \ldots d\,\theta_{N-2}\, d\,\phi \qquad (5)$$

Metric Tensor for R^N

The metric tensor for R^N in spherical coordinates is defined as

$$\gamma_{i\,j} = \sum_{k=1}^{N} \frac{\partial\, x_k}{\partial\, u_i} \frac{\partial\, x_k}{\partial\, u_j} \qquad i,\ j = 1,\ \ldots,\ N \quad \text{with}\ \ u_i = \theta_i \ \text{for}\ i = 1,\ \ldots,\ N-1,\ u_{N-1} = \phi\ \ \&$$

$u_N = r$

Hence,

$$\gamma_{i\,j} = g_{i\,j} \qquad \text{for} \qquad i,\ j = 1,\ \ldots,\ N-1$$

From eq(1), we have

$$\frac{\partial\, x_k}{\partial\, r} = \frac{x_k}{r} \qquad \forall\ k$$

(6a)

Using eq(1), we have, for $i = 1,\ N-1$,

$$\gamma_{i,\,N} = \sum_{k=i}^{N} \frac{\partial\, x_k}{\partial\, \theta_i} \frac{\partial\, x_k}{\partial\, r} = \frac{1}{r} \sum_{k=i}^{N} \frac{\partial\, x_k}{\partial\, \theta_i} x_k = \frac{1}{2\,r} \frac{\partial}{\partial\, \theta_i} \sum_{k=i}^{N} x_k^2$$

$$= \frac{r}{2} \frac{\partial}{\partial\, \theta_i} \prod_{k=1}^{i-1} \sin^2 \theta_k \qquad \text{[Eq(T1) used]}$$

$$= 0$$

(6b)

Hence, $\gamma_{i\,j}$ is also diagonal.

$$\gamma_{N,\,N} = \sum_{k=1}^{N} \left(\frac{\partial\, x_k}{\partial\, r} \right)^2 = \frac{1}{r^2} \sum_{k=1}^{N} x_k^2 = 1$$

(6c)

Hence, $\gamma_{i\,j}$ can be treated as an $N \times N$ matrix $\boldsymbol{\gamma}$.

$$\boldsymbol{\gamma} = \gamma_{i\,j} = r^2 \operatorname{diag}\left(1,\ \sin^2 \theta_1,\ \sin^2 \theta_1 \sin^2 \theta_2,\ \ldots,\ \ \sin^2 \theta_1 \ldots \sin^2 \theta_{N-2}\,,\ \frac{1}{r^2} \right)$$

(6)

$$\boldsymbol{\gamma}^{-1} = \gamma^{i\,j} = \frac{1}{r^2} \operatorname{diag}\left(1,\ \frac{1}{\sin^2 \theta_1},\ \frac{1}{\sin^2 \theta_1 \sin^2 \theta_2},\ \ldots,\ \frac{1}{\sin^2 \theta_1 \ldots \sin^2 \theta_{N-2}},\ r^2 \right)$$

$$\therefore \qquad \gamma = \det \boldsymbol{\gamma}\ = g\ = r^{2(N-1)} \prod_{j=1}^{N-2} \sin^{2(N-j-1)} \theta_j$$

(6d)

Laplacian

The Laplacian in a space described by coordinates $\{u^i\}$ & metric tensor g_{ij} is given by (see Frankel, § 2.9c)

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial u^j}\right)$$

(7)

For the spherical coordinates in R^N, we have

$$\nabla^2 f = \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{N} \frac{\partial}{\partial u^i}\left(\sqrt{\gamma}\, \gamma^{ii} \frac{\partial f}{\partial u^i}\right)$$

$$= \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial r}\left(\sqrt{\gamma} \frac{\partial f}{\partial r}\right) + \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{N-1} \frac{\partial}{\partial u^i}\left(\sqrt{\gamma}\, \gamma^{ii} \frac{\partial f}{\partial u^i}\right)$$

Using eq(6d), we obtain the Laplace-Beltrami operator

$$\nabla^2 f = \frac{1}{r^{N-1}} \frac{\partial}{\partial r}\left(r^{N-1} \frac{\partial f}{\partial r}\right) + \frac{1}{r^2} \nabla^2_{S^{N-1}} f$$

(8)

where

$$\nabla^2_{S^{N-1}} f = \frac{1}{\sqrt{\gamma}} \sum_{i=1}^{N-1} \frac{\partial}{\partial u^i}\left(\sqrt{\gamma}\, r^2 \gamma^{ii} \frac{\partial f}{\partial u^i}\right)$$

(8a)

is the spherical Laplacian.

For $i = 1, \ ..., N-1$,

$$r^2 \gamma^{ii} = \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_k}$$

$$\frac{\partial}{\partial \theta^i}\left(\sqrt{\gamma}\, X\right) = \frac{\sqrt{\gamma}}{\sin^{N-i-1} \theta_i} \frac{\partial}{\partial \theta^i}\left(\sin^{N-i-1} \theta_i X\right) \qquad \text{[Eq(6) used]}$$

the i^{th} term in the sum in $\nabla^2_{S^{N-1}} f$ is therefore

$$\mathcal{I}_i = \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_k} \frac{1}{\sin^{N-i-1} \theta_i} \frac{\partial}{\partial \theta_i}\left(\sin^{N-i-1} \theta_i \frac{\partial f}{\partial \theta_i}\right)$$

$$\therefore \qquad \nabla^2_{S^{N-1}} f = \sum_{i=1}^{N-1} \frac{1}{\prod_{k=1}^{i-1} \sin^2 \theta_k} \frac{1}{\sin^{N-i-1} \theta_i} \frac{\partial}{\partial \theta_i}\left(\sin^{N-i-1} \theta_i \frac{\partial f}{\partial \theta_i}\right)$$

(9)

$$= \frac{1}{\sin^{N-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{N-2}\theta_1\frac{\partial f}{\partial\theta_1}\right)$$
$$+\frac{1}{\sin^2\theta_1}\frac{1}{\sin^{N-3}\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin^{N-3}\theta_2\frac{\partial f}{\partial\theta_2}\right)$$
$$+\frac{1}{\sin^2\theta_1\sin^2\theta_2}\frac{1}{\sin^{N-4}\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^{N-4}\theta_3\frac{\partial f}{\partial\theta_3}\right)$$
$$+\ldots$$
$$+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{j-1}}\frac{1}{\sin^{N-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{N-j-1}\theta_j\frac{\partial f}{\partial\theta_j}\right)$$
$$+\ldots$$
$$+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{N-3}}\frac{1}{\sin\theta_{N-2}}\frac{\partial}{\partial\theta_{N-2}}\sin\theta_{N-2}\frac{\partial f}{\partial\theta_{N-2}}$$
$$+\frac{1}{\sin^2\theta_1\ldots\sin^2\theta_{N-2}}\frac{\partial^2 f}{\partial\phi^2}$$

(9a)

$$= \frac{1}{\sin^{N-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{N-2}\theta_1\frac{\partial f}{\partial\theta_1}\right) \quad (9b)$$
$$+\frac{1}{\sin^2\theta_1}\left\{\frac{1}{\sin^{N-3}\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin^{N-3}\theta_2\frac{\partial f}{\partial\theta_2}\right)\right.$$
$$+\frac{1}{\sin^2\theta_2}\left[\frac{1}{\sin^{N-4}\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^{N-4}\theta_3\frac{\partial f}{\partial\theta_3}\right)\right.$$
$$+\ldots$$
$$+\frac{1}{\sin^2\theta_{j-1}}\left[\frac{1}{\sin^{N-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{N-j-1}\theta_j\frac{\partial f}{\partial\theta_j}\right)+\ldots\right.$$
$$\left.\left.\left.+\frac{1}{\sin^2\theta_{N-3}}\left[\frac{1}{\sin\theta_{N-2}}\frac{\partial}{\partial\theta_{N-2}}\left(\sin\theta_{N-2}\frac{\partial f}{\partial\theta_{N-2}}\right)+\frac{1}{\sin^2\theta_{N-2}}\frac{\partial^2 f}{\partial\phi^2}\right]\right]\ldots\right]\right\}$$

Let $\theta = \theta_{N-n-1}$, $\xi^n = \{\theta_{N-n}, .., \theta_{N-1}\}$ & set

$$\nabla^2_{\xi^{n+1}} f = \nabla^2_{(\theta,\xi^n)} f = \frac{1}{\sin^n\theta}\frac{\partial}{\partial\theta}\left(\sin^n\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\nabla^2_{\xi^n} f \qquad \forall\ n = N-2,\ \ldots,\ 1$$

(10)

with $\nabla^2_{\xi^1} f = \dfrac{\partial^2 f}{\partial \phi^2}$.

Then eq(9b) can be obtained recursively using eq(10) & starting from $n = N-2$ down to $n = 1$:

$$\nabla^2_{S^{N-1}} f = \nabla^2_{\xi^{N-1}} f = \frac{1}{\sin^{N-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{N-2}\theta_1 \frac{\partial f}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}\nabla^2_{\xi^{N-2}} f$$

$$= \frac{1}{\sin^{N-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{N-2}\theta_1 \frac{\partial f}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}\left[\frac{1}{\sin^{N-3}\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin^{N-3}\theta_2 \frac{\partial f}{\partial\theta_2}\right) + \frac{1}{\sin^2\theta_2}\nabla^2_{\xi^{N-3}} f\right]$$

$$= \frac{1}{\sin^{N-2}\theta_1}\frac{\partial}{\partial\theta_1}\left(\sin^{N-2}\theta_1 \frac{\partial f}{\partial\theta_1}\right) + \frac{1}{\sin^2\theta_1}\left\{\frac{1}{\sin^{N-3}\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin^{N-3}\theta_2 \frac{\partial f}{\partial\theta_2}\right)\right.$$

$$\left. + \frac{1}{\sin^2\theta_2}\left[\frac{1}{\sin^{N-4}\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^{N-4}\theta_3 \frac{\partial f}{\partial\theta_3}\right) + \frac{1}{\sin^2\theta_3}\nabla^2_{\xi^{N-4}} f\right]\right\}$$

$\vdots$

At the final iteration, $n = 1$, one gets $\theta = \theta_{N-2}$ so that

$$\nabla^2_{\xi^2} f = \frac{1}{\sin\theta_{N-2}}\frac{\partial}{\partial\theta_{N-2}}\left(\theta_{N-2}\frac{\partial f}{\partial\theta_{N-2}}\right) + \frac{1}{\sin^2\theta_{N-2}}\frac{\partial^2 f}{\partial\phi^2}$$

in agreement with eq(9b).

Helmholtz Eq.

The N-D Helmholtz eq.

$$\nabla^2 f = -k^2 f$$

(11)

is separable in spherical coordinates

$$(r, \boldsymbol{\theta}) = (r, \theta_1, \ldots, \theta_{N-1}) = (r, \theta_1, \ldots, \theta_{N-2}, \phi)$$

To begin, we set

$$f = R(r)\, Y(\boldsymbol{\theta}) = R(r)\prod_{i=1}^{N-1} Y_i(\theta_i) \qquad (11a)$$

$$\to \quad \frac{1}{f}\nabla^2 f = \frac{1}{R\, r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial R}{\partial r}\right) + \frac{1}{r^2\, Y}\nabla^2_{S^{N-1}} Y = -k^2$$

Setting

$$\frac{1}{Y}\nabla^2_{S^{N-1}} Y = \alpha_{N-1} = \text{const}$$

(11b)

gives the radial eq.

$$\frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{dR}{dr}\right) + \left(k^2 + \frac{\alpha_{N-1}}{r^2}\right)R = 0$$

(11c)

Eq(10) indicates that $\nabla^2_{S^{N-1}}$ is separable since, with $\theta = \theta_{N-n-1}$, we have

$$\frac{1}{Y}\nabla^2_{\xi^{n+1}} Y = \frac{1}{Y_{N-n-1}\sin^n\theta}\frac{d}{d\theta}\left(\sin^n\theta\frac{d Y_{N-n-1}}{d\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{1}{Y}\nabla^2_{\xi^n} Y\right)$$

(12a)

Thus, setting

$$\frac{1}{Y}\nabla^2_{\xi^n} Y = \alpha_n = \text{const} \tag{12b}$$

we have, for $n = 1, \ldots, N-2$,

$$\frac{1}{\sin^n\theta}\frac{d}{d\theta}\left(\sin^n\theta\,\frac{dY_{N-n-1}}{d\theta}\right) + \left(\frac{\alpha_n}{\sin^2\theta} - \alpha_{n+1}\right)Y_{N-n-1} = 0 \tag{13}$$

&

$$\frac{d^2 Y_{N-1}}{d\phi^2} = \alpha_1 Y_{N-1}$$

(13a)

Eq(13a) is easily solved. With the boundary condition

$$Y_{N-1}(\phi) = Y_{N-1}(\phi + 2\pi)$$

we have

$$Y_{N-1}(\phi) \propto e^{\pm i m \phi} \qquad \text{with} \qquad \alpha_1 = -m^2 \qquad \& \qquad m \in \mathbb{Z} \tag{14}$$

For $n = 1$, eq(13) becomes

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dY_{N-2}}{d\theta}\right) + \left(-\frac{m^2}{\sin^2\theta} - \alpha_2\right)Y_{N-2} = 0$$

which is just the eq. for the associated Legendre functions $P_l^m(\cos\theta)$ if we set

$$\alpha_2 = -l\,(l+1) \qquad l = 0, 1, 2, \ldots \qquad \& \qquad m = -l, \ldots, l$$

(Reminder, solutions without the restrictions on l & m are singular.)

Solutions to eq(13) for $n > 1$ are therefore some generalization of P_l^m.

Case $\alpha_n = 0 \;\forall\; n = 1, \ldots, N-3$

If one is only interested in the radial eq(11c) or the eigenvalues k^2, then the only relevant information from the angular part is α_{N-1}. Since any rotation can be accomplished by a single rotation about some axis, we can always choose the coordinate axes such that only θ_{N-2} is non-zero. Thus, to obtain α_{N-1}, we can set $\alpha_n = 0 \;\forall\; n = 1, \ldots N-2$ without affecting the values of k^2.

Setting

$$\alpha_n = 0 \qquad \forall \quad n = 1, \ldots, N-2$$

eq(13) becomes

$$\frac{1}{\sin^n\theta}\frac{d}{d\theta}\left(\sin^n\theta\,\frac{dY_{N-n-1}}{d\theta}\right) = 0 \qquad \forall \quad n = 1, \ldots, N-3$$

(15a)

$$\frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\left(\sin^{N-2}\theta\,\frac{dY_1}{d\theta}\right) - \alpha_{N-1} Y_1 = 0 \tag{15}$$

Setting $x = \cos\theta$, we have

$$dx = -\sin\theta\, d\theta$$

$$\to \quad \frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\,\frac{d}{dx}$$

so that

$$\frac{1}{\sin^n\theta}\frac{d}{d\theta}\left(\sin^n\theta\,\frac{dY}{d\theta}\right) = \frac{1}{(1-x^2)^{(n-1)/2}}\frac{d}{dx}\left((1-x^2)^{(n+1)/2}\frac{dY}{dx}\right)$$

$$= (1 - x^2) \frac{d^2 Y}{d x^2} - (n + 1) x \frac{d Y}{d x}$$

(15b)

Comparing with the ultraspherical equation

$$(1 - x^2) \frac{d^2 C_l^{(\alpha)}}{d x^2} - (2 \alpha + 1) x \frac{d C_l^{(\alpha)}}{d x} + l (l + 2 \alpha) C_l^{(\alpha)} = 0$$

(15c)

where $C_l^{\alpha}(x)$ are the Gegenbauer (or ultraspherical) polynomials, we see that by setting

$$\alpha = \frac{n}{2} \qquad \& \qquad \alpha_{N-1} = -l\,(l + 2\,\alpha) = -l\,(l + N - 2)$$

(16)

we have

$$Y_{N-n-1}(\cos\theta) = C_0^{n/2}(\cos\theta) = 1 \qquad \forall \quad n = 1, \ldots, N - 3$$

(16a)

$$Y_1(\cos\theta) = C_l^{(N-2)/2}(\cos\theta)$$

(16b)

so that eq(11b) becomes

$$\nabla^2_{S^{N-1}} Y(\boldsymbol{\theta}) = -l\,(l + N - 2)\, Y(\boldsymbol{\theta}) \qquad l = 0, 1, 2, \ldots$$

(16c)

As mentioned in the openning of this section, although eq(16c) was derived explicitly for the special case $Y(\boldsymbol{\theta}) = Y_1(\cos\theta_1)$, it should hold for any general solution $Y(\boldsymbol{\theta})$.