4.1. The Langevin Equation

Langevin equation is a first order in time stochastic differential equation of the form

$$\dot{q}_i(t) = -\frac{1}{2} f_i[\boldsymbol{q}(t)] + v_i(t) \qquad \left(\dot{q}_i = \frac{d q_i}{d t} \right) \tag{4.1}$$

where $\boldsymbol{q} \in R^d$ & $\boldsymbol{v}$ is a set of d stochastic functions called "noise" with a functional probability distribution $[d \rho(\boldsymbol{v})]$, which we shall assume to be gaussian.

$$[d \rho(\boldsymbol{v})] = [d \boldsymbol{v}] \exp\left[-\frac{1}{2\Omega} \int_{-\infty}^{\infty} d t \sum_{i=1}^{d} v_i^2(t) \right] \tag{4.2}$$

where Ω is the width of the noise.

Dividing the interval $(-T, T)$ into $2N$ segments of length ε, we have $\varepsilon = T/N$. Eq(4.2) can be written in discretized form as

$$[d \rho(\boldsymbol{v})] = [d \boldsymbol{v}] \exp\left[-\frac{\varepsilon}{2\Omega} \sum_{i=1}^{d} \sum_{k=-N}^{N} v_i^2(t_k) \right] \qquad (T, N \to \infty)$$

$$= [d \boldsymbol{v}] \prod_{i=1}^{d} \prod_{k=-N}^{N} \exp\left[-\frac{\varepsilon}{2\Omega} v_i^2(t_k) \right]$$

where

$$t_k = k \varepsilon$$

Using

$$\int_{-\infty}^{\infty} d x \exp\left(-\frac{1}{2} a x^2 \right) = \sqrt{\frac{2\pi}{a}}$$

we see that the normalization

$$1 = \int [d \rho(\boldsymbol{v})] = C \prod_{i=1}^{d} \left(\int \prod_{k=-N}^{N-1} d v_i(t_k) \exp\left[-\frac{\varepsilon}{2\Omega} v_i^2(t_k) \right] \right)$$

requires

$$[d v] = \left(\frac{\varepsilon}{2\pi\Omega} \right)^{dN} \prod_{i=1}^{d} \int \prod_{k=-N}^{N-1} d v_i(t_k) \tag{4.2a}$$

with $N \to \infty$ & $\varepsilon \to 0$ under stood.

This particular form of the gaussian noise, called gaussian white noise, is related to Markov's processes (see Appendix A4).

Note that Ω can be absorbed as follows. Setting

$$\tau = \Omega t \qquad w_i(\tau) = \frac{1}{\Omega} v_i(t) \qquad F_i = \frac{f_i}{\Omega}$$

we have

$$[d \rho(\boldsymbol{w})] = [d \boldsymbol{w}] \exp\left[-\frac{1}{2} \int d\tau \sum_i w_i^2(\tau) \right]$$

& $$q_i'(\tau) = -\frac{1}{2} F_i[\boldsymbol{q}(\tau)] + w_i(\tau) \qquad \left(q_i' = \frac{d q_i}{d \tau} \right)$$

Gaussian noise can also be characterized by its 1- & 2-point correlation functions:

$$\langle v_i(t) \rangle = 0 \qquad \langle v_i(t)\, v_j(t') \rangle = \Omega\, \delta_{ij}\, \delta(t - t') \tag{4.3}$$

Proof:

Using eq(4.2a), we have

$$\langle v_i(t)\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right)^{dN} \prod_{m=1}^{d} \int_{-\infty}^{\infty} \prod_{k=-N}^{N-1} d\,v_m(t_k) \left(v_i(t) \exp\left[-\frac{\varepsilon}{2\Omega} v_m^2(t_k)\right]\right)$$

For $m \neq i$, each integral gives a factor $\left(\frac{2\pi\Omega}{\varepsilon}\right)^{1/2}$ so that we're left with

$$\langle v_i(t)\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right)^{N} \int_{-\infty}^{\infty} \prod_{k=-N}^{N-1} d\,v_i(t_k) \left(v_i(t) \exp\left[-\frac{\varepsilon}{2\Omega} v_i^2(t_k)\right]\right)$$

Let $t = t_p$, then for all $k \neq p$, each integral gives a factor $\left(\frac{2\pi\Omega}{\varepsilon}\right)^{1/2}$ & we have

$$\langle v_i(t)\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right)^{1/2} \int_{-\infty}^{\infty} d\,v_i(t_p) \left(v_i(t_p) \exp\left[-\frac{\varepsilon}{2\Omega} v_i^2(t_p)\right]\right)$$

Using

$$\int_{-\infty}^{\infty} d\,x\,x \exp\left(-\frac{1}{2} a x^2\right) = 0$$

we have

$$\langle v_i(t)\rangle = 0$$

thus proving the 1st part of eq(4.3).

Similarly,

$$\langle v_i(t)\, v_j(t')\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right)^{dN} \prod_{m=1}^{d} \int_{-\infty}^{\infty} \prod_{k=-N}^{N-1} d\,v_m(t_k) \left(v_i(t)\, v_j(t') \exp\left[-\frac{\varepsilon}{2\Omega} v_m^2(t_k)\right]\right)$$

For $m \neq i$ or j, each integral gives a factor $\left(\frac{2\pi\Omega}{\varepsilon}\right)^{1/2}$ so that we're left with

$$\langle v_i(t)\, v_j(t')\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right)^{2N} \int_{-\infty}^{\infty} \prod_{k=-N}^{N-1} d\,v_i(t_k)\, d\,v_j(t_k) \left(v_i(t)\, v_j(t') \exp\left[-\frac{\varepsilon}{2\Omega} \left(v_i^2(t_k) + v_j^2(t_k)\right)\right]\right)$$

Let $t = t_p$ & $t' = t_q$, then for all $k \neq p$ or q, each integral gives a factor $\left(\frac{2\pi\Omega}{\varepsilon}\right)^{1/2}$ & we have

$$\langle v_i(t)\, v_j(t')\rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right) \int_{-\infty}^{\infty} d\,v_i(t_p)\, d\,v_j(t_q) \left(v_i(t_p)\, v_j(t_q) \exp\left[-\frac{\varepsilon}{2\Omega} \left(v_i^2(t_p) + v_j^2(t_q)\right)\right]\right)$$

Using

$$\int_{-\infty}^{\infty} d\,x\,x \exp\left(-\frac{1}{2} a x^2\right) = 0 \qquad\qquad \int_{-\infty}^{\infty} d\,x\,x^2 \exp\left(-\frac{1}{2} a x^2\right) = \sqrt{\frac{2\pi}{a^3}}$$

we have

$$\begin{aligned}
\langle v_i(t)\, v_j(t')\rangle &= \delta_{ij}\, \delta_{pq} \left(\frac{\varepsilon}{2\pi\Omega}\right)^{1/2} \int_{-\infty}^{\infty} d\,v_i(t_p)\, v_i(t_p)^2 \exp\left[-\frac{\varepsilon}{2\Omega} v_i^2(t_p)\right] \\
&= \delta_{ij}\, \delta_{pq} \frac{\Omega}{\varepsilon} \\
&= \delta_{ij} \begin{cases} \frac{\Omega}{\varepsilon} & \text{for } -\frac{\varepsilon}{2} < t - t' < \frac{\varepsilon}{2} \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{ij} \begin{cases} \frac{N\Omega}{T} & \text{for } -\frac{T}{2N\Omega} < \frac{t-t'}{\Omega} < \frac{T}{2N\Omega} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Using [see,e.g., G.B.Arfken, “Mathematical Methods for Physicists”, (index: delta function, sequence)]

$$\delta_n(x) = \begin{cases} n & \text{for } -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases} \qquad \rightarrow \qquad \lim_{n\to\infty} \delta_n(x) = \delta(x)$$

we have

$$\langle v_i(t)\, v_j(t') \rangle = \delta_{ij}\, \delta\left(\frac{t-t'}{\Omega}\right) = \delta_{ij}\, \Omega\, \delta(t-t')$$

which is the 2nd part of eq(4.3).

Note: One should go over the above derivation to convince oneself that one can use the gaussian integral properties to write down immediately, without going through the intermediate steps, that

$$\langle v_i(t)\, v_j(t') \rangle = \left(\frac{\varepsilon}{2\pi\Omega}\right) \int_{-\infty}^{\infty} d\, v_i(t)\, d\, v_j(t') \left(v_i(t)\, v_j(t') \exp\left[-\frac{\varepsilon}{2\Omega}\left(v_i^2(t) + v_j^2(t')\right)\right]\right) \qquad (4.3a)$$

Given $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$, the probability distribution of finding $\boldsymbol{q}(t) = \boldsymbol{q}$ is defined as

$$P(\boldsymbol{q}, t;\, \boldsymbol{q}_0, t_0) = \left\langle \prod_{i=1}^{d} \delta[\, q_i(t) - q_i\,] \right\rangle_v \qquad (\, t \geq t_0\,) \qquad (4.4)$$

where the subscript v denotes the gaussian distribution eq(4.2) and $\boldsymbol{q}(t)$ obeys the Langevin eq(4.4).

For any function $O(\boldsymbol{q})$,

$$\langle O[\boldsymbol{q}(t)] \rangle_v = \int d\, \boldsymbol{q}\, P(\boldsymbol{q}, t;\, \boldsymbol{q}_0, t_0)\, O(\boldsymbol{q})$$

(4.5)

Since the Langevin eq. is local in time, P is Markovian with

$$P(\boldsymbol{q}_3, t_3;\, \boldsymbol{q}_1, t_1) = \int d\, \boldsymbol{q}_2\, P(\boldsymbol{q}_3, t_3;\, \boldsymbol{q}_2, t_2)\, P(\boldsymbol{q}_2, t_2;\, \boldsymbol{q}_1, t_1) \qquad (\, t_3 \geq t_2 \geq t_1\,) \qquad (4.6)$$

In terms of the bra-ket notation, we define

$$P(\boldsymbol{q}, t;\, \boldsymbol{q}_0, t_0) \equiv \langle\, \boldsymbol{q} \mid P(t, t_0) \mid \boldsymbol{q}_0 \rangle \qquad (4.7)$$

Since eq(4.4) is invariant under time translation,

$$P(\boldsymbol{q}, t;\, \boldsymbol{q}_0, t_0) = P(\, \boldsymbol{q}, \boldsymbol{q}_0\, ;\, t - t_0\,)$$

$\rightarrow \qquad P(t, t_0) = P(t - t_0)$

Hence, there exists an operator H such that

$$P(t, t_0) = e^{-(t-t_0)\, H}$$

(4.8)

H is known as the Fokker-Planck hamiltonian.

Note: Sometimes we’ll omit the initial data & simplify the notation by writing $P(\boldsymbol{q}, t;\, \boldsymbol{q}_0, t_0)$ simply as $P(\boldsymbol{q}, t)$.