4.2. A Simple Example: The Linear Langevin Equation

Let

$$\dot{q} = -\omega\, q + v(t) \tag{4.9}$$

where v is the gaussian noise [eq(4.3)] with $\Omega = 1$.

The homogeneous solution is

$$q_h(t) = c\, e^{-\omega t}$$

which is also the reciprocal of the integrating factor.

The particular soution is [see Arfken],

$$q_p(t) = q_h(t) \int^t d\,s\, \frac{v(s)}{q_h(s)}$$

$$= e^{-\omega t} \int^t d\,s\; v(s)\, e^{\omega s}$$

Given the initial condition

$$q(0) = q_0$$

we have

$$q(t) = q_0\, e^{-\omega t} + \int_0^t d\,s\; v(s)\, e^{-\omega (t-s)} \tag{4.10}$$

Random Walk

Case $\omega = 0$ corresponds to a simple random walk in the continuum.

Eq(4.10) becomes

$$q(t) = q_0 + \int_0^t d\,s\; v(s) \tag{4.10a}$$

Using eq(4.3), we have

$$\langle\, q(t)\,\rangle_v = q_0 + \int_0^t d\,s\, \langle\, v(s)\,\rangle_v$$

$$= q_0$$

Eq(4.10a) gives

$$\left\langle [\, q(t) - q_0\,]^2 \right\rangle_v = \left\langle\, \int_0^t d\,s\; v(s) \int_0^t d\,u\; v(u) \right\rangle_v$$

$$= \int_0^t d\,s \int_0^t d\,u\, \langle\, v(s)\, v(u)\,\rangle_v$$

Using eq(4.3), we have

$$\left\langle [\, q(t) - q_0\,]^2 \right\rangle_v = \int_0^t d\,s \int_0^t d\,u\, \delta(s-u)$$

$$= \int_0^t d\,s$$

$$= t \tag{4.10b}$$

Eq(4.10) can also be written as

$$q(t+\varepsilon) - q(t) = \int_t^{t+\varepsilon} d\,s\; v(s)$$

$$\rightarrow \qquad \left\langle [\, q(t+\varepsilon) - q(t)\,]^2 \right\rangle_v = \left\langle\, \int_t^{t+\varepsilon} d\,s\; v(s) \int_t^{t+\varepsilon} d\,u\; v(u) \right\rangle_v$$

$$= \int_t^{t+\varepsilon} ds \int_t^{t+\varepsilon} du\ \delta(s-u)$$

$$= \varepsilon$$

$$\therefore \qquad |\ q(t+\varepsilon) - q(t)\ | \underset{\varepsilon\to 0}{=} O\left(\sqrt{\varepsilon}\right) \qquad (4.11)$$

Hence, the derivative

$$\lim_{\varepsilon\to 0} \frac{q(t+\varepsilon) - q(t)}{\varepsilon} = O\left(\varepsilon^{-1/2}\right)$$

is not defined. The path of a random walk (or Brownian motion) is not differentiable. $\dot{q}$ in the Langevin eq. is therefore rather symbolic.

General Case

Taking the average of eq(4.10) gives

$$\langle q(t) \rangle_v = q_0\, e^{-\omega t} + \int_0^t ds\, \langle v(s) \rangle\, e^{-\omega (t-s)}$$

$$= q_0\, e^{-\omega t} \qquad (4.12)$$

Eq(4.10) thus becomes

$$q(t) = \langle q(t) \rangle_v + \int_0^t ds\, v(s)\, e^{-\omega (t-s)} \qquad (4.12a)$$

$$\to \qquad \left\langle [\, q(t) - \langle q(t) \rangle]^2 \right\rangle_v = \left\langle \int_0^t ds\, v(s)\, e^{-\omega (t-s)} \int_0^t du\, v(u)\, e^{-\omega (t-u)} \right\rangle_v$$

$$= e^{-2\omega t} \int_0^t ds \int_0^t du\, e^{\omega (s+u)}\, \delta(s-u) \quad [\text{ eq(4.3) used }]$$

$$= e^{-2\omega t} \int_0^t ds\, e^{2\omega s}$$

$$= \frac{e^{-2\omega t}}{2\omega} \left(e^{2\omega t} - 1 \right)$$

$$= \frac{1}{2\omega} \left(1 - e^{-2\omega t} \right) \qquad (4.13)$$

Eqs(4.12-3) show that the distribution of q converges (diverges) as $t \to \infty$ for $\omega > (<)\, 0$.

Since $q(t)$ is linearly related to $v(t)$, it too has a gaussian distribution $P(q, t)$ characterized by its two first moments $\langle q(t) \rangle_v$ & $\left\langle q(t)^2 \right\rangle_v$.

Using eq(4.4) &

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx}$$

we have

$$P(q, t) = \langle \delta[\, q(t) - q\,] \rangle_v$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \langle \exp(\, ik\, [\, q(t) - q\,]\,) \rangle_v$$

Eqs(4.12 & 12a) give

$$q(t) - q = q_0\, e^{-\omega t} - q + \int_0^t ds\, v(s)\, e^{-\omega (t-s)}$$

$$\rightarrow \qquad P(q,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(q_0 e^{-\omega t} - q)} \left\langle \exp\left(ik \int_0^t ds\, v(s)\, e^{-\omega(t-s)} \right) \right\rangle_v$$

$$\mathcal{I} = \left\langle \exp\left(ik \int_0^t ds\, v(s)\, e^{-\omega(t-s)} \right) \right\rangle_v$$

$$= \int [dv(u)] \exp\left(ik \int_0^t ds\, v(s)\, e^{-\omega(t-s)} - \frac{1}{2}\int_{-\infty}^{\infty} du\, v(u)^2 \right)$$

$$= \int [dv(s)] \exp\left(\int_0^t ds \left[ik\, v(s)\, e^{-\omega(t-s)} - \frac{1}{2} v(s)^2 \right] \right)$$

where every factor involving $v(u)$ with $u \notin [0, t]$ has been integrated to give 1.

As in §4.1, after discretizing, the $v(s)$ integral becomes

$$\left(\frac{\varepsilon}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dv(s) \exp\left(\varepsilon \left[ik\, v(s)\, e^{-\omega(t-s)} - \frac{1}{2} v(s)^2 \right] \right) = \exp\left(-\frac{1}{2} \varepsilon k^2 e^{-2\omega(t-s)} \right)$$

where the gaussian integral formula

$$\int_{-\infty}^{\infty} dx\, e^{-a x^2 + b x} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

was used.

Hence,

$$\mathcal{I} = \prod_{s=0}^{t/N} \exp\left(-\frac{1}{2} \varepsilon k^2 e^{-2\omega(t-s)} \right)$$

$$= \exp\left(-\sum_{s=0}^{t/N} \frac{1}{2} \varepsilon k^2 e^{-2\omega(t-s)} \right)$$

$$= \exp\left(-\frac{1}{2} k^2 \int_0^t ds\, e^{-2\omega(t-s)} \right)$$

$$= \exp\left(-\frac{k^2}{4\omega} (1 - e^{-2\omega t}) \right)$$

$$\rightarrow \qquad P(q,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp\left(ik (q_0 e^{-\omega t} - q) - \frac{k^2}{4\omega}(1 - e^{-2\omega t}) \right)$$

$$= \sqrt{\frac{\omega}{\pi(1 - e^{-2\omega t})}} \exp\left(-\frac{\omega}{1 - e^{-2\omega t}} (q_0 e^{-\omega t} - q)^2 \right) \qquad (4.14)$$

Note that

$$\int_{-\infty}^{\infty} dq\, P(q,t) = 1$$

as befits a probability distribution.

Assuming $\omega > 0$, the equilibrium distribution is

$$P(q) = P(q, t \to \infty)$$

$$= \sqrt{\frac{\omega}{\pi}} \exp(-\omega q^2) \qquad (4.15)$$

Remark

For $q_0 = 0$, eqs(4.12 & 4.12a) give

$$q(t) = \int_0^t d\,s\, v(s)\, e^{-\omega(t-s)}$$

The $q(t)$ 2-point function is therefore

$$\langle q(t_1)\, q(t_2) \rangle_v = \int_0^{t_1} d\,s\, e^{-\omega(t_1-s)} \int_0^{t_2} d\,u\, e^{-\omega(t_2-u)} \langle v(s)\, v(u) \rangle_v$$

$$= \int_0^{t_1} d\,s\, e^{-\omega(t_1-s)} \int_0^{t_2} d\,u\, e^{-\omega(t_2-u)}\, \delta(s-u) \qquad \text{[eq(4.3) used]}$$

For $t_1 > t_2$, $\delta(s-u) = 0$ for $s > t_2$. Hence,

$$\langle q(t_1)\, q(t_2) \rangle_v = e^{-\omega(t_1+t_2)} \int_0^{t_2} d\,s\, e^{2\omega s} = \frac{e^{-\omega(t_1+t_2)}}{2\omega} \left(e^{2\omega t_2} - 1 \right)$$

$$= \frac{1}{2\omega} \left(e^{-\omega(t_1-t_2)} - e^{-\omega(t_1+t_2)} \right)$$

For $t_1 < t_2$, $\delta(s-u) = 0$ for $u > t_1$. Hence,

$$\langle q(t_1)\, q(t_2) \rangle_v = e^{-\omega(t_1+t_2)} \int_0^{t_1} d\,u\, e^{2\omega u} = \frac{e^{-\omega(t_1+t_2)}}{2\omega} \left(e^{2\omega t_1} - 1 \right)$$

$$= \frac{1}{2\omega} \left(e^{-\omega(t_2-t_1)} - e^{-\omega(t_1+t_2)} \right)$$

Combining the 2 results, we have

$$\langle q(t_1)\, q(t_2) \rangle_v = \frac{1}{2\omega} \left(e^{-\omega|t_1-t_2|} - e^{-\omega(t_1+t_2)} \right) \qquad (4.16)$$

Differentiating with respect to t_1, we find

$$\left\langle \dot{q}(t_1)\, q(t_2) \right\rangle_v = \frac{1}{2}\left[-\epsilon(t_1-t_2)\, e^{-\omega|t_1-t_2|} + e^{-\omega(t_1+t_2)} \right] \qquad (4.17)$$

where

$$\epsilon(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

is the sign function.

Now,

$$\left\langle \dot{q}(t_1)\, q(t_2) \right\rangle_v = \begin{cases} \frac{1}{2}\left(-1 + e^{-2\omega t_1} \right) & \text{for } t_1 - t_2 = 0_+ \\ \frac{1}{2}\left(1 + e^{-2\omega t_1} \right) & \text{for } t_1 - t_2 = 0_- \end{cases} \qquad (4.17a)$$

so that $\left\langle \dot{q}(t)\, q(t) \right\rangle_v$ is undefined.

However, from eq(4.16), we have

$$\left\langle q(t)^2 \right\rangle_v = \frac{1}{2\omega}\left(1 - e^{-2\omega t} \right)$$

$$\rightarrow \qquad \frac{d}{d\,t} \langle q(t)\, q(t) \rangle_v = e^{-2\omega t}$$

while eq(4.17a) says $\left\langle \frac{d\,q(t)}{d\,t}\, q(t) \right\rangle_v$ is undefined.

Thus, in general, time differentiation and averaging do not commute:

$$\frac{d}{dt}\langle ... \rangle \neq \left\langle \frac{d}{dt} ... \right\rangle$$

However, setting $\epsilon(0) = 0$ & eq(4.17) becomes

$$\left\langle \frac{d\,q(t)}{d\,t}\,q(t) \right\rangle_{\nu} = \frac{1}{2}\,e^{-2\,\omega\,t}$$

so that time differentiation and averaging commute:

$$\frac{d}{dt}\langle ... \rangle = \left\langle \frac{d}{dt} ... \right\rangle$$

This is reminiscent of the situation encountered in §3.2.