4.8. Brownian Motion on Riemannian Manifolds

We now discuss the general Langevin equation directly in the continuum limit.

Consider the general markovian Langevin equation:

$$\dot{q}^i(t) = -\frac{1}{2} f^i[q(t)] + e^i_a[q(t)]\, \nu_a(t) \tag{4.63}$$

where $\nu_a(t)$ is a gaussian white noise with

$$\langle \nu_a(t) \rangle = 0 \qquad \langle \nu_a(t)\, \nu_b(t') \rangle = \Omega\, \delta(t-t')\, \delta_{ab} \tag{4.64}$$

Integrating eq(4.63) with $q^i(t) = q'^i$ gives

$$q^i(t+\varepsilon) = q'^i - \frac{1}{2} \int_t^{t+\varepsilon} d\tau\, f^i[\,q(\tau)] + \int_t^{t+\varepsilon} d\tau\, e^i_a[q(\tau)]\, \nu_a(\tau) \tag{1}$$

which can be evaluated by iteration using

$$q^i(\tau) = q'^i - \frac{1}{2} \int_t^{\tau} d\tau'\, f^i[\,q(\tau')] + \int_t^{\tau} d\tau'\, e^i_a[q(\tau')]\, \nu_a(\tau') \tag{2}$$

Replacing $q(\tau)$ with q', we obtain the lowest order of eq(1):

$$q^i(t+\varepsilon) = q'^i - \frac{1}{2}\varepsilon f^i(q') + e^i_a(q') \int_t^{t+\varepsilon} d\tau'\, \nu_a(\tau')$$

Since

$$\int_t^{t+\varepsilon} d\tau'\, \nu_a(\tau') = O\left(\sqrt{\varepsilon}\right)$$

we have

$$q^i(t+\varepsilon) = q'^i + e^i_a(q') \int_t^{t+\varepsilon} d\tau'\, \nu_a(\tau') + O(\varepsilon) \tag{3}$$

$$\rightarrow \qquad \int_t^{t+\varepsilon} d\tau\, f^i[\,q(\tau)] = \int_t^{t+\varepsilon} d\tau\, \left\{ f^i(q') + \left[\, q^j(\tau) - q'^j \,\right] \partial_j f^i(q') + \ldots \right\}$$

$$= \varepsilon f^i(q') + O\left(\varepsilon^{3/2}\right)$$

Thus, to $O(\varepsilon)$, we have

$$q^i(t+\varepsilon) = q'^i - \frac{1}{2}\varepsilon f^i(q') + \int_t^{t+\varepsilon} d\tau \left\{ e^i_a(q') + \left[\, q^j(\tau) - q'^j \,\right] \partial_j e^i_a(q') + \ldots \right\} \nu_a(\tau)$$

$$= q'^i - \frac{1}{2}\varepsilon f^i(q') + \int_t^{t+\varepsilon} d\tau\, e^i_a(q')\, \nu_a(\tau) + e^j_b(q')\, \partial_j e^i_a(q') \int_t^{t+\varepsilon} d\tau \int_t^{\tau} d\tau'\, \nu_b(\tau')\, \nu_a(\tau) + O\left(\varepsilon^{3/2}\right)$$

The term quadratic in ν can be replaced by its average [see §4.6].

$$\int_t^{t+\varepsilon} d\tau \int_t^{\tau} d\tau' \langle \nu_b(\tau')\, \nu_a(\tau) \rangle = \int_t^{t+\varepsilon} d\tau \int_t^{\tau} d\tau'\, \Omega\, \delta(\tau-\tau')\, \delta_{ab} \qquad \text{[Eq(4.64) used]}$$

Since $\delta(\tau - \tau') \neq 0$ at the edge of the domain of the τ' integral, there is ambiguouity which, in the discretized version of the problem, amounts to deciding the value of the step function

$$\theta(\tau) = \frac{1}{2}[\,1 + \epsilon(\tau)\,] = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

at $\tau = 0$. If we follow the choice made in §3.2.1, namely, $\epsilon(0) = 0$, we have $\theta(0) = \frac{1}{2}$ so that

$$\int_t^{t+\varepsilon} d\tau \int_t^{\tau} d\tau' \langle \nu_b(\tau')\, \nu_a(\tau) \rangle = \Omega\, \delta_{ab} \int_t^{t+\varepsilon} d\tau\, \theta(0)$$

$$= \Omega\,\delta_{a\,b}\,\varepsilon\,\theta(0)$$
$$= \frac{1}{2}\,\Omega\,\delta_{a\,b}\,\varepsilon$$

$$\to \qquad q^i(t+\varepsilon) = q'^{\,i} - \frac{1}{2}\,\varepsilon\, f^i(q') + e^i_a(q')\int_t^{t+\varepsilon} d\,\tau\, v_a(\tau) + \frac{1}{2}\,\Omega\,\varepsilon\, e^j_a(q')\,\partial_j e^i_a(q') + O\left(\varepsilon^{3/2}\right) \qquad (4.65)$$

Setting

$$\int_t^{t+\varepsilon} d\,\tau\; v_a(\tau) = \overline{v}_a(t)$$

we have

$$q^i(t+\varepsilon) = q'^{\,i} - \frac{1}{2}\,\varepsilon\, f^i(q') + e^i_a(q')\,\overline{v}_a(t) + \frac{1}{2}\,\Omega\,\varepsilon\, e^j_a(q')\,\partial_j e^i_a(q') + O\left(\varepsilon^{3/2}\right) \qquad (4.66)$$

Discussion

The ambiguity in eq(4.63) has the same nature of the problem of operator ordering discussed in §3.2.2 [see eq(3.20)]. As in §4.7, the problem is better understood in discretized form.

If we replace $\boldsymbol{q}(t)$ with the symmetric form $\frac{1}{2}[\,\boldsymbol{q}(t+\varepsilon)+\boldsymbol{q}(t)\,]$, the term $e^i_a[\boldsymbol{q}(t)]$ in eq(4.63) becomes

$$e^i_a\left\{\frac{1}{2}[\,\boldsymbol{q}(t+\varepsilon)+\boldsymbol{q}(t)\,]\right\} = e^i_a[\boldsymbol{q}(t)] + \frac{1}{2}\left[\,q^j\,(t+\varepsilon) - q^j\,(t)\,\right]\partial_j\, e^i_a[\boldsymbol{q}\,(t)] + \ldots$$
$$= e^i_a[\boldsymbol{q}(t)] + \frac{1}{2}\, e^j_b[\boldsymbol{q}(t)]\, v_b(t)\,\partial_j\, e^i_a[\boldsymbol{q}\,(t)] + O(\varepsilon) \qquad (4.67)$$

Eq(4.63) is then just eq(4.52) with

$$d^i_{a\,b}(\boldsymbol{q}) = e^j_b(\boldsymbol{q})\,\partial_j\, e^i_a(\boldsymbol{q}) \qquad (4.68)$$

The choice $\varepsilon(0) = 0$, & hence eq(4.65), thus corresponds to the Stratanovich convention of using the symmetric form $\frac{1}{2}[\,\boldsymbol{q}(t+\varepsilon)+\boldsymbol{q}(t)\,]$. We shall adopt this convention since it has simpler properties under a change of coordinates.

4.8.1. The Fokker–Planck Equation

Using eq(4.68) on eqs(4.56 & 4.60) of §4.7, we obtain

$$h^i(\boldsymbol{q}) = f^i(\boldsymbol{q}) - \Omega\, d^i_{a\,a}(\boldsymbol{q})$$
$$= f^i(\boldsymbol{q}) - \Omega\, e^j_a(\boldsymbol{q})\,\partial_j e^i_a(\boldsymbol{q})$$

$$H = \frac{1}{2}\,\hat{p}_i\left\{\Omega\,\hat{p}_j\, g^{i\,j}(\hat{\boldsymbol{q}}) - i\, h^i(\hat{\boldsymbol{q}})\right\} \qquad \text{with} \qquad g^{i\,j}(\boldsymbol{q}) = e^i_a\,(\boldsymbol{q})\, e^j_a\,(\boldsymbol{q})$$
$$= \frac{1}{2}\,\hat{p}_i\left\{\Omega\,\hat{p}_j\, e^i_a\,(\hat{\boldsymbol{q}})\, e^j_a\,(\hat{\boldsymbol{q}}) - i\, f^i\,(\hat{\boldsymbol{q}}) + i\,\Omega\, e^j_a(\hat{\boldsymbol{q}})\,\partial_j e^i_a(\hat{\boldsymbol{q}})\right\} \qquad (4.69a)$$

The FP eq. for $P(\boldsymbol{q},\, t)$ is therefore [see eq(4.23 & 4.61)]

$$\frac{\partial}{\partial\, t}P(\boldsymbol{q},\, t) = -H\, P(\boldsymbol{q},\, t)$$
$$= \frac{1}{2}\,\partial_i\left\{\Omega\,\partial_j\left[\, e^j_a(\boldsymbol{q})\, e^i_a(\boldsymbol{q})\, P\right] + f^i(\boldsymbol{q})\, P - \Omega\, e^j_a(\boldsymbol{q})\left[\,\partial_j e^i_a(\boldsymbol{q})\right] P\right\}$$
$$= \frac{1}{2}\,\partial_i\left\{\Omega\, e^i_a(\boldsymbol{q})\,\partial_j\left[\, e^j_a(\boldsymbol{q})\, P\right] + f^i(\boldsymbol{q})\, P\right\} \qquad (4.70)$$

Brownian Motion on a Riemannian Manifold

The divergence of a vector $\boldsymbol{V}$ on a Riemannian manifold is defined as [see n-Sphere.pdf]

$$\nabla\cdot\boldsymbol{V} = \frac{1}{2\sqrt{g}}\partial_i\left[\sqrt{g}\,V^i\right] = \frac{1}{2\sqrt{g}}\partial_i\left[\sqrt{g}\,g^{ij}\,V_j\right] \qquad (4.71a)$$

where $g_{ij}(\boldsymbol{q})$ is the metric tensor, $g_{ij}\,g^{jk} = \delta_i^k$, & $g = \det g_{ij}$.

The FP eq for free Brownian motion is therefore

$$\dot{D}(q, t) = \frac{\Omega}{2\sqrt{g}}\partial_i\left(\sqrt{g}\,g^{ij}\,\partial_j D\right) \qquad (4.71)$$

where $D(q, t)$ is a scalar density with normalization

$$\int d\,\boldsymbol{q}\,\sqrt{g}\,D = 1 \qquad (4.71b)$$

As in §3.2, D has a path intgeral representation

$$D = \int\left[d\,\boldsymbol{q}(\tau)\,\sqrt{g}\,\right]e^{-S(\boldsymbol{q})/\Omega} \qquad (4.72)$$

with

$$S(\boldsymbol{q}) = \frac{1}{2}\int d\,\tau\,g_{ij}\,\dot{q}^i\,\dot{q}^j \qquad (4.73)$$

In general,

$$\sqrt{g}\,D(\boldsymbol{q}, t) = P(\boldsymbol{q}, t) \qquad (4.74)$$

This also implies that if the matrix e_a^i is a square matrix, it is the inverse of the vielbein (tetrads) [see §22.6]

For an introduction to the bundle of linear frames, see R.Aldrovandi & J.G.Pereira, "An Introduction to Geometrical Physics", §9.3.

If the manifold is compact (finite & in 1 piece with no holes), a constant scalar density D is obviously a static solution to eq(4.71) and, therefore, corresponds to the equilibrium distribution.

Putting eq(4.74) into eq(4.70), we have

$$\frac{\partial}{\partial t}\left(\sqrt{g}\,D\right) = \frac{1}{2}\partial_i\left\{\Omega\,e_a^i\,\partial_j\left(e_a^j\sqrt{g}\,D\right) + f^i\sqrt{g}\,D\right\}$$

$$\rightarrow \qquad \dot{D} = \frac{1}{2\sqrt{g}}\partial_i\left\{\Omega\,e_a^i\,\partial_j\left(e_a^j\sqrt{g}\,D\right) + f^i\sqrt{g}\,D\right\} \qquad (4)$$

Whereas, the FP eq. for free Brownian motion is [see eqs(4.69 & 71)]

$$\dot{D}_0 = \frac{\Omega}{2\sqrt{g}}\partial_i\left(\sqrt{g}\,e_a^i\,e_a^j\,\partial_j D_0\right)$$

Eq(4) can be written as

$$\dot{D} = \dot{D}_0 + \frac{\Omega}{2\sqrt{g}}\partial_i\left\{e_a^i\,D\,\partial_j\left(e_a^j\sqrt{g}\right) + \frac{1}{\Omega}f^i\sqrt{g}\,D\right\}$$

Therefore, $D = D_0$ if

$$f^i = -\Omega\,\frac{e_a^i}{\sqrt{g}}\partial_j\left(e_a^j\sqrt{g}\right)$$

$$= -\Omega\,e_a^i\left(\partial_j e_a^j + \frac{1}{2g}e_a^j\,\partial_j g\right)$$

$$= -\Omega\left(e_a^i \partial_j e_a^j + \frac{1}{2g} g^{jj} \partial_j g \right)$$

Using [see eq(d) in §"Covariant Derivative"]

$$\Gamma^j_{jk} g^{ki} = \frac{1}{2g} g^{ki} \partial_k g$$

where Γ^i_{jk} is the metric connection of the manifold, we have

$$f^i = -\Omega\left(e_a^i \partial_j e_a^j + \Gamma^j_{jk} g^{ki} \right)$$

$$= -\Omega\, e_a^i \left(\partial_j e_a^j + \Gamma^j_{jk} e_a^k \right)$$

$$= -\Omega\, e_a^i \nabla_j e_a^j \qquad \text{[eq(c) used]} \qquad (4.75)$$

4.8.2. Path Integral Representation

Applying the method of §4.6 to eq(4.57), we can derive a path integral representation for the probability distribution, and correlation functions.

In the following, a more direct derivation based on the Langevin eq(4.63) is given.

Integral Representation of Constraints

Consider the mapping $\boldsymbol{x} \mapsto \boldsymbol{y}$ given by

$$y_i = f_i(\boldsymbol{x}) \qquad (4.76)$$

where $f_i(\boldsymbol{x})$ are differentiable functions & eq(4.76) is invertible locally, which means the jacobian

$$J(\boldsymbol{x}) = \left| \det \frac{\partial f_i}{\partial x_j} \right| \qquad (4.77)$$

exits everywhere.

We want to calculate a function $\sigma(\boldsymbol{x})$ for all $\boldsymbol{x}$ that satisfy the constraint equation $\boldsymbol{f}(\boldsymbol{x}) = 0$, without solving the equation explicitly. This can be done formally using the δ-function:

$$\sigma(\boldsymbol{x}) \mid_{f(\boldsymbol{x})=0} = \int d\boldsymbol{x}\, \delta(\boldsymbol{x} - \boldsymbol{\mu})\, \sigma(\boldsymbol{x})$$

$$= \int \left\{ \prod_{i=1}^{n} d x_i\, \delta(x_i - \mu_i) \right\} \sigma(\boldsymbol{x})$$

where $\boldsymbol{\mu}$ is a solution of $\boldsymbol{f}(\boldsymbol{x}) = 0$.

Using eq(g) of § "$\delta[\boldsymbol{f}(\boldsymbol{x})]$", we have

$$\sigma(\boldsymbol{x}) \mid_{f(\boldsymbol{x})=0} = \int d\boldsymbol{x}\, \delta[f(\boldsymbol{x})]\, J(\boldsymbol{x})\, \sigma(\boldsymbol{x})$$

$$= \int \left\{ \prod_{i=1}^{n} d x_i\, \delta[f_i(\boldsymbol{x})] \right\} J(\boldsymbol{x})\, \sigma(\boldsymbol{x}) \qquad (4.78)$$

Using

$$\delta[f(\boldsymbol{x})] = \int \frac{d\boldsymbol{k}}{(2\pi)^n} e^{i\boldsymbol{k}\cdot\boldsymbol{f}}$$

we have

$$\sigma(\boldsymbol{x}) \mid_{f(\boldsymbol{x})=0} = \int d\boldsymbol{x} \frac{d\boldsymbol{k}}{(2\pi)^n} e^{i\boldsymbol{k}\cdot\boldsymbol{f}(\boldsymbol{x})} J(\boldsymbol{x})\, \sigma(\boldsymbol{x})$$

$$= \frac{1}{(2\pi)^n} \int \left\{ \prod_{i=1}^{n} d x_i \, d k_i \right\} e^{i \boldsymbol{k} \cdot \boldsymbol{f}(\boldsymbol{x})} J(\boldsymbol{x}) \, \sigma(\boldsymbol{x}) \tag{4.79}$$

Path Integral Representation

We want to find the path integral representation of

$$P(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) = \langle \delta[\boldsymbol{q}(t) - \boldsymbol{q}] \rangle_v$$

where $\boldsymbol{q}(\tau)$ for $\tau \in [t, t_0]$ obeys the Langevin eq(4.63), which can be written as

$$F^i[\boldsymbol{q}(\tau)] = \dot{q}^i(\tau) + \frac{1}{2} f^i[q(\tau)] - e^i_a[q(\tau)] \, v_a(\tau) = 0 \tag{4.80a}$$

Since there is a solution to eq(4.80a) for each $\tau \in [t, t_0]$, eq(4.79) leads to

$$P(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) = \int_{\boldsymbol{q}(t_0) = \boldsymbol{q}_0}^{\boldsymbol{q}(t) = \boldsymbol{q}} [d \, \boldsymbol{q}(\tau)] \int \frac{d \boldsymbol{k}}{(2\pi)^n} \left\langle e^{i \boldsymbol{k} \cdot \boldsymbol{F}[\boldsymbol{q}(\tau)]} J(\boldsymbol{q}) \right\rangle_v \tag{4.80b}$$

J

Since $F^j[\boldsymbol{q}(\tau)]$ depends on 2 indices i & τ, the corresponding jacobian “matrix” has 4 indices. The jacobian J [see eq(4.77)] is therefore

$$J(\boldsymbol{q}) = \left| \det \frac{\delta F^j[\boldsymbol{q}(t)]}{\delta q^i(\tau)} \right| \qquad [\det \boldsymbol{A} = \det \boldsymbol{A}^T \text{ used }]$$

$$\equiv \left| \det \tilde{M}^j_i(t, \tau) \right|$$

where $\frac{\delta}{\delta q}$ is the functional derivative.

Using

$$\frac{\delta q^j(t)}{\delta q^i(\tau)} = \delta^j_i \, \delta(t - \tau)$$

$$\frac{\delta X[q(t)]}{\delta q^i(\tau)} = \delta(t - \tau) \left. \frac{\partial X(q)}{\partial q^i} \right|_{q = q(\tau)}$$

$$\equiv \delta(t - \tau) \, \partial_i X[q(\tau)]$$

we have

$$\tilde{M}^j_i(t, \tau) = \delta^j_i \frac{d}{d t} \delta(t - \tau) + \frac{1}{2} \delta(t - \tau) \, \partial_i f^j[q(\tau)] - \delta(t - \tau) \, \partial_i e^j_a[q(\tau)] \, v_a(\tau) \tag{4.80}$$

Using

$$\delta(t - \tau) = \frac{d}{d t} \theta(t - \tau) \qquad \text{where} \qquad \theta(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

eq(4.80) becomes

$$\tilde{M}^j_i(t, \tau) = \frac{d}{d t} \left\{ \delta^j_i \, \delta(t - \tau) + \theta(t - \tau) \left[\frac{1}{2} \partial_i f^j[q(\tau)] - \partial_i e^j_a[q(\tau)] \, v_a(\tau) \right] \right\}$$

$$= \frac{d}{d t} M^j_i(t, \tau)$$

where

$$M^j_i(t, \tau) = \delta^j_i \, \delta(t - \tau) + \theta(t - \tau) \left\{ \frac{1}{2} \partial_i f^j[q(\tau)] - \partial_i e^j_a[q(\tau)] \, v_a(\tau) \right\}$$

$$= \delta_i^j\, \delta(t-\tau) + \theta(t-\tau)\, v_i^j(\tau) \tag{4.81}$$

with

$$v_i^j(\tau) = \frac{1}{2}\,\partial_i f^j[q(\tau)] - \partial_i e_a^j[q(\tau)]\, v_a(\tau) \tag{4.82}$$

Using [see eq(1.101)]

$$\det \boldsymbol{A} = e^{\mathrm{tr}\ln \boldsymbol{A}}$$

we have

$$\ln\det \boldsymbol{M} = \mathrm{tr}\ln \boldsymbol{M} = \int d\,t\,(\ln \boldsymbol{M})_i^i\,(t,\,t)$$

$$= \int d\,t\,\{\ln[\,\delta(0)\,\boldsymbol{I} + \theta(0)\,\boldsymbol{v}(t)]\,\}_i^i$$

Using

$$\ln(1+\boldsymbol{X}) = \sum_{n=1}^{\infty} (-)^{n+1}\,\frac{\boldsymbol{X}^n}{n}$$

with

$$\boldsymbol{X}(\,t,\,i\,;\;\tau,\,j\,) = \theta(t-\tau)\,v_i^j(\tau)$$

$$\boldsymbol{X}^2\,(\,t,\,i\,;\;\tau,\,j\,) = \int d\,t_1\,\theta\,(t-t_1)\,\theta\,(t_1-\tau)\,v_i^k\,(t_1)\,v_k^j(\tau)$$

$$\boldsymbol{X}^n\,(\,t,\,i\,;\;\tau,\,j\,) = \int d\,t_1 \ldots d\,t_{n-1}\,\theta\,(t-t_1)\ldots\theta\,(t_{n-1}-\tau)\,v_i^{k_1}\,(t_1)\ldots v_{k_{n-1}}^j(\tau)$$

we have

$$\ln\det \boldsymbol{M} = \int d\,t\,\theta(0)\,v_i^i(t) - \frac{1}{2}\int d\,t\,d\,t_1\,\theta\,(t-t_1)\,\theta\,(t_1-t)\,v_i^k\,(t_1)\,v_k^i(t)$$

$$+ \ldots + \frac{(-)^{n+1}}{n}\int d\,t\,d\,t_1 \ldots d\,t_{n-1}\;\;\theta\,(t-t_1)\ldots\theta\,(t_{n-1}-t)\,v_i^{k_1}\,(t_1)\ldots v_{k_{n-1}}^i(t)$$

$$= \theta(0)\int d\,t\;\;\mathrm{tr}\,\boldsymbol{v}(t) - \frac{1}{2}\int d\,t\,d\,t_1\,\theta\,(t-t_1)\,\theta(t_1-t)\,\mathrm{tr}[\boldsymbol{v}\,(t_1)\,\boldsymbol{v}(t)] \tag{4.83}$$

$$+ \ldots + \frac{(-)^{n+1}}{n}\int d\,t\,d\,t_1 \ldots d\,t_{n-1}\;\;\theta\,(t-t_1)\ldots\theta(t_{n-1}-t)\,\mathrm{tr}[\boldsymbol{v}(t_1)\ldots\boldsymbol{v}(t)]$$

Since

$$\theta\,(t-t_1)\ldots\theta(t_{n-1}-t_n) \neq 0 \qquad \text{only if} \quad t > t_1 \ldots > t_n$$

we have

$$\theta\,(t-t_1)\ldots\theta(t_{n-1}-t) = 0 \qquad \forall\;\, n \geq 2$$

Hence,

$$\ln\det \boldsymbol{M} = \theta(0)\int d\,t\;\;\mathrm{tr}\,\boldsymbol{v}(t)$$

$$\rightarrow \qquad \det \boldsymbol{M} = \exp\left(\theta(0)\int d\,t\;\;\mathrm{tr}\,\boldsymbol{v}(t)\right) \tag{4.84}$$

As usual, we'll set $\theta(0) = \frac{1}{2}$.

From eq(4.82), we have

$$\det \boldsymbol{M} = \exp\left[\frac{1}{2}\int d\,t\left(\frac{1}{2}\,\partial_i f^i - \partial_i e_a^i\, v_a\right)\right]$$

$$J = \det \dot{\boldsymbol{M}} = \frac{d}{d\,t}\det \boldsymbol{M} \;\propto \exp\left[\frac{1}{2}\int d\,t\left(\frac{1}{2}\,\partial_i f^i - \partial_i e_a^i\, v_a\right)\right] \tag{4.85}$$

The Noise Average

Using eq(4.80a), we have, for each τ,

$$\delta[\boldsymbol{F}(\boldsymbol{q})] = \int \frac{d\,\boldsymbol{k}}{(2\,\pi)^n}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{F}}$$

$$= \int d\,\boldsymbol{\lambda}(\tau)\, e^{\boldsymbol{\lambda}\cdot\boldsymbol{F}} \qquad\qquad \boldsymbol{\lambda} = i\,\boldsymbol{k}$$

$$= \int d\,\boldsymbol{\lambda}(\tau)\, \exp\left\{\lambda_i\left[\dot{q}^i + \frac{1}{2} f^i(q) - e^i_a(q)\, v_a\right]\right\} \qquad (4.85a)$$

The v_a integral (or noise average) in eq(4.80b) includes contributions coming from J, $\delta[\boldsymbol{F}(\boldsymbol{q})]$ & the gaussian distribution of the noise. Using eqs(4.85,4.85a), we have, for each τ,

$$\mathcal{I}_v = \int d\,\boldsymbol{v}(\tau)\, \exp\left[-\left(\frac{1}{2}\partial_i e^i_a\, v_a + \lambda_i\, e^i_a\, v_a + \frac{1}{2\,\Omega} v_a^2\right)\right]$$

Using eq(1.8) with

$$b = \frac{1}{2}\partial_i e^i_a + \lambda_i\, e^i_a \qquad\qquad A = \frac{1}{\Omega}$$

we have

$$\mathcal{I}_v \propto \exp\left[\frac{1}{2}\Omega\left(\frac{1}{2}\partial_i e^i_a + \lambda_i\, e^i_a\right)^2\right]$$

$$\left(\frac{1}{2}\partial_i e^i_a + \lambda_i\, e^i_a\right)^2 = \frac{1}{4}\partial_i e^i_a\, \partial_j e^j_a + \lambda_i\, e^i_a\, \partial_j e^j_a + \lambda_i\, e^i_a\, \lambda_j\, e^j_a \qquad\qquad g^{i\,j} = e^i_a\, e^j_a$$

$$= \frac{1}{4}\partial_i e^i_a\, \partial_j e^j_a + \lambda_i\, e^i_a\, \partial_j e^j_a + \lambda_i\, \lambda_j\, g^{i\,j} \qquad (4.85b)$$

The $\boldsymbol{\lambda}$ integral, eq(4.85a), thus becomes, for each τ,

$$\mathcal{I}_\lambda = \int d\,\boldsymbol{\lambda}(\tau)\, \exp\left\{\lambda_i\left(\dot{q}^i + \frac{1}{2} f^i\right) + \frac{1}{2}\Omega\left(\lambda_i\, e^i_a\, \partial_j e^j_a + \lambda_i\, \lambda_j\, g^{i\,j}\right)\right\}$$

Using eq(1.8) with

$$b_i = \dot{q}^i + \frac{1}{2} f^i + \frac{1}{2}\Omega\, e^i_a\, \partial_j e^j_a \qquad\qquad A_{i\,j} = \Omega\, g^{i\,j}$$

we have

$$\mathcal{I}_\lambda \propto \sqrt{g}\; e^{-s_1(\boldsymbol{q})/\Omega}$$

where (∂ operates on nearest neighbor only)

$$s_1(\boldsymbol{q}) = \frac{1}{2}\left(\dot{q}^i + \frac{1}{2} f^i + \frac{1}{2}\Omega\, e^i_a\, \partial_k e^k_a\right) g_{i\,j}\left(\dot{q}^j + \frac{1}{2} f^j + \frac{1}{2}\Omega\, e^j_b\, \partial_m e^m_b\right)$$

$$= \frac{1}{2}\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{i\,j}\left(\dot{q}^j + \frac{1}{2} f^j\right) + \frac{1}{2}\Omega\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{i\,j}\, e^j_a\, \partial_m e^m_a + \frac{1}{8}\Omega^2\, e^i_a\, \partial_k e^k_a\, g_{i\,j}\, e^j_b\, \partial_m e^m_b$$

$$= \frac{1}{2}\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{i\,j}\left(\dot{q}^j + \frac{1}{2} f^j\right) + \frac{1}{2}\Omega\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{a\,i}\, \partial_m e^m_a + \frac{1}{8}\Omega^2\, e^i_a\, \partial_k e^k_a\, e_{b\,i}\, \partial_m e^m_b$$

If we now assume that e^i_a is a square invertible matrix (and, therefore, plays the role of a vielbein), we have

$$e^i_a\, e_{b\,i} = \delta_{a\,b} \qquad\qquad e_{a\,i}\, e^m_a = \delta^m_i$$

$$\rightarrow \qquad s_1(q) = \frac{1}{2}\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{i\,j}\left(\dot{q}^j + \frac{1}{2} f^j\right) + \frac{1}{2}\Omega\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{a\,i}\, \partial_m e^m_a + \frac{1}{8}\Omega^2\, \partial_k e^k_a\, \partial_m e^m_a$$

Hence, for each τ,

$$\mathcal{I}_\nu \mathcal{I}_\lambda \propto \sqrt{g}\; e^{-s(\boldsymbol{q})/\Omega}$$

with

$$s(\boldsymbol{q}) = \frac{1}{2}\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{ij} \left(\dot{q}^j + \frac{1}{2} f^j\right) + \frac{1}{2}\Omega\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{ai}\, \partial_m e_a^m$$

Sweeping all constants into $[\,d\,q\,]$, we have

$$P(\boldsymbol{q}, t; \boldsymbol{q}_0, t_0) = \int_{\boldsymbol{q}(t_0)=\boldsymbol{q}_0}^{\boldsymbol{q}(t)=\boldsymbol{q}} \left[\, d\,q(\tau) \sqrt{g[\boldsymbol{q}(\tau)]}\,\right] e^{-\mathcal{S}(\boldsymbol{q})/\Omega} \tag{4.86}$$

where

$$\begin{aligned}\mathcal{S}(\boldsymbol{q}) &= \int_{t_0}^{t} d\,\tau\, s[\boldsymbol{q}(\tau)] \\ &= \frac{1}{2}\int_{t_0}^{t} d\,\tau \left[\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{ij} \left(\dot{q}^j + \frac{1}{2} f^j\right) + \Omega\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{ai}\, \partial_m e_a^m\right] \qquad (4.85c)\end{aligned}$$

Eq(4.85c) can be re-written in a covariant form by replacing ∂_i with the covariant derivative ∇_i as follows.

$$\begin{aligned}\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{ai}\, \partial_m e_a^m &= -\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\, \partial_m e_{ai} \qquad [\, e_{ai}\, e_a^m = \delta_i^m \text{ used }] \\ &= -\partial_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_{ai}\, e_a^m\right] + e_{ai}\, \partial_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] \\ &= -\partial_i\left(\dot{q}^i + \frac{1}{2} f^i\right) + e_{ai}\, \partial_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right]\end{aligned}$$

Using [see eqs(c & f) of §"Covariant Derivative"],

$$\begin{aligned}\nabla_j\left(u^i w^j\right) &= \partial_j\left(u^i w^j\right) + \Gamma^i_{jk}\, u^k w^j + \Gamma^j_{jk}\, u^i w^k \\ &= \partial_j\left(u^i w^j\right) + \Gamma^i_{jk}\, u^k w^j + (\partial_k \ln e)\, u^i w^k\end{aligned}$$

we have

$$\begin{aligned}e_{ai}\, \partial_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] &= e_{ai}\left\{\nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] - (\partial_m \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m - \Gamma^i_{mk}\left(\dot{q}^k + \frac{1}{2} f^k\right) e_a^m\right\} \\ &= e_{ai}\, \nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] - (\partial_i \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right) - (\partial_k \ln e)\left(\dot{q}^k + \frac{1}{2} f^k\right) \\ &= e_{ai}\, \nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] - 2\,(\partial_i \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right)\end{aligned}$$

$$\begin{aligned}\rightarrow \quad \left(\dot{q}^i + \frac{1}{2} f^i\right) e_{ai}\, \partial_m e_a^m &= -\partial_i\left(\dot{q}^i + \frac{1}{2} f^i\right) + e_{ai}\, \nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] - 2\,(\partial_i \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right) \\ &= -\nabla_i\left(\dot{q}^i + \frac{1}{2} f^i\right) + e_{ai}\, \nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] - (\partial_i \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right)\end{aligned}$$

Using

$$\int_{t_0}^{t} d\,\tau\, \nabla_i \dot{q}^i = \int_{t_0}^{t} d\,\tau \frac{d}{d\,\tau} \nabla_i q^i = \nabla_i q^i \Big|_{\boldsymbol{q}(t_0)}^{\boldsymbol{q}(t)} = 0$$

eq(4.85c) becomes

$$\mathcal{S}(\boldsymbol{q}) = \frac{1}{2}\int_{t_0}^{t} d\,\tau \left\{\left(\dot{q}^i + \frac{1}{2} f^i\right) g_{ij} \left(\dot{q}^j + \frac{1}{2} f^j\right) - \frac{1}{2}\Omega\, \nabla_i f^i + \Omega\, e_{ai}\, \nabla_m\left[\left(\dot{q}^i + \frac{1}{2} f^i\right) e_a^m\right] + A\right\}$$

(4.87a)

which differs from Zinn-Justin's eq(4.87) by the term

$$A = -(\partial_i \ln e)\left(\dot{q}^i + \frac{1}{2} f^i\right)$$

Eq(4.87a) is not covariant due to the presence of A.

Appendix

$\partial_k(\det A)$

Consider a square matrix $A = (a_{ij})$. Let $C(i, j)$ be the matrix obtained by striking out the i^{th} row & j^{th} column of A. The cofactor C^{ij} of a_{ij} is defined as

$$C^{ij} = (-)^{i+j} \det C(i, j) \qquad \text{(a)}$$

The inverse $A^{-1} = (a^{ij})$ is given by

$$a^{ij} = \frac{1}{a} C^{ji}$$

where

$$a = \det A$$

The Laplace expansion of a gives

$$a = \sum_j a_{ij} C^{ij} \qquad (\text{ no summation on } i\ )$$

$$= \sum_i a_{ij} C^{ij} \qquad (\text{ no summation on } j\ )$$

By the definition eq(a), C^{ij} does not contain a_{ij}. Therefore

$$\frac{\partial a}{\partial a_{ij}} = C^{ij}$$

$$\to \qquad \partial_k a = \frac{\partial a}{\partial a_{ij}} \partial_k a_{ij} = C^{ij} \partial_k a_{ij}$$

$$= a\, a^{ji} \partial_k a_{ij}$$

$$a^{ji} a_{ij} = \delta^j_j = d \qquad \text{for } A \text{ an } d \times d \text{ matrix.}$$

$$\to \qquad a^{ji} \partial_k a_{ij} = -a_{ij} \partial_k a^{ji}$$

$$\partial_k a = -a\, a_{ij} \partial_k a^{ji} \qquad \text{(b)}$$

Covariant Derivative

Ref: Any book on general relativity.
A readable introductory text is
I.D.Lawrie, “A Unified Grand Tour of Theoretical Physics”, 2nd ed., §2.3.

When operating on a vector $\boldsymbol{u}$, or 1-form $\boldsymbol{w}$, the covariant derivative ∇ is defined as

$$\nabla_j u^i = \partial_j u^i + \Gamma^i_{jk} u^k \qquad \text{(c)}$$

$$\nabla_j w_i = \partial_j w_i - \Gamma^k_{ji} w_k$$

where, for a Riemannian manifold, Γ^i_{jk} is the metric connection related to the metric tensor g_{ij} by

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} \right)$$

Using

$$0 = \partial_j \delta^i_k = \partial_j \left(g^{im} g_{mk} \right) = g^{im} \partial_j g_{mk} + g_{mk} \partial_j g^{im}$$

$$\partial_m \left(g^{im} g_{jk} \right) = g^{im} \partial_m g_{jk} + g_{jk} \partial_m g^{im}$$

we have

$$\Gamma^i_{jk} = -\frac{1}{2} \left(g_{mk} \partial_j + g_{mj} \partial_k - g_{jk} \partial_m \right) g^{im} - \frac{1}{2} \partial_m \left(g^{im} g_{jk} \right)$$

$$\Gamma^j_{jk} g^{ki} = -\frac{1}{2} \left(\delta^i_m \partial_j + g_{mj} g^{ki} \partial_k - \delta^i_j \partial_m \right) g^{jm} - \frac{1}{2} g^{ki} \partial_m \left(g^{jm} g_{jk} \right)$$

$$= -\frac{1}{2} \left(\partial_j g^{ji} + g_{mj} g^{ki} \partial_k g^{jm} - \partial_m g^{im} \right)$$

$$= -\frac{1}{2} g_{mj} g^{ki} \partial_k g^{jm}$$

$$= \frac{1}{2g} g^{ki} \partial_k g \qquad \text{[Eq(b) used]} \qquad \text{(d)}$$

Let

$e = \det \mathbf{e} = \det e_{ai}$ [e_{ai} treated as the (a, i) element of matrix **e**.]

then

$$e^2 = \det \mathbf{e}^T \det \mathbf{e} = \det \left(\mathbf{e}^T \mathbf{e} \right) = \det \left(e_{ia} e_{aj} \right) = \det g_{ij} = g$$

$$\rightarrow \qquad e = \sqrt{g} \qquad \text{(e)}$$

Eq(d) in §“Covariant Derivative” can be written in terms of e^i_a as

$$e^i_a \left(\Gamma^j_{jk} e^k_a - \frac{1}{2g} e^k_a \partial_k g \right) = 0$$

If **e** is invertible ($e \neq 0$), then its rows or columns (e^i_a) are linearly independent. Hence,

$$\Gamma^j_{jk} = \frac{1}{2g} \partial_k g = \frac{1}{2} \partial_k \ln g = \partial_k \ln e$$

Eq(c) thus implies

$$\nabla_i u^i = \partial_i u^i + \Gamma^i_{ik} u^k$$

$$= \partial_i u^i + (\partial_k \ln e) u^k \qquad \text{(f)}$$

$$\nabla_m (e_{ai} e^m_a) = 0 = e_{ai} \nabla_m e^m_a + e^m_a \nabla_m e_{ai}$$

$$= e_{ai} [\partial_m e^m_a + (\partial_m \ln e) e^m_a] + e^m_a \left(\partial_m e_{ai} - \Gamma^k_{mi} e_{ak} \right)$$

$$= \partial_i \ln e - \Gamma^k_{ki}$$

$\delta[\boldsymbol{f}(\boldsymbol{x})]$

Let $\boldsymbol{\mu}$ be a solution to $\boldsymbol{f}(\boldsymbol{x}) = 0$, i.e., $f(\boldsymbol{\mu}) = 0$. Then for $\boldsymbol{x}$ near $\boldsymbol{\mu}$, we have

$$f_i(\boldsymbol{x}) = (x_j - \mu_j) \frac{\partial f_i(\boldsymbol{\mu})}{\partial x_j} + \ldots$$

$$\rightarrow \qquad \int d\boldsymbol{x}\, \delta[f(\boldsymbol{x})] = \int \prod_{i=1}^{n} d x_i\, \delta\!\left[(x_j - \mu_j) \frac{\partial f_i(\boldsymbol{\mu})}{\partial x_j} \right]$$

Let

$$z_i = (x_j - \mu_j)\,\frac{\partial f_i(\boldsymbol{\mu})}{\partial x_j}$$

$$\rightarrow \qquad \frac{\partial z_i}{\partial x_j} = \frac{\partial f_i(\boldsymbol{\mu})}{\partial x_j}$$

For each root $\boldsymbol{\mu}$ of $\boldsymbol{f}$,

$$1 = \int_{-\boldsymbol{\epsilon}}^{\boldsymbol{\epsilon}} \prod_{i=1}^{n} d\,z_i\;\delta(z_i) = \int_{\boldsymbol{\lambda}-\epsilon}^{\boldsymbol{\lambda}+\boldsymbol{\epsilon}} J(\boldsymbol{\mu}) \prod_{i=1}^{n} d\,x_i\,\delta(z_i) \qquad J(\boldsymbol{\mu}) = \left|\det\left(\frac{\partial z_i}{\partial x_j}\right)\right|$$

$$= \int_{\boldsymbol{\lambda}-\boldsymbol{\epsilon}}^{\boldsymbol{\lambda}+\boldsymbol{\epsilon}} \prod_{i=1}^{n} d\,x_i\;\delta(x_i - \mu_i)$$

Since $\boldsymbol{f} = \boldsymbol{z}$ near each $\boldsymbol{\lambda}$, we have

$$\delta(\boldsymbol{f}) = \sum_{\boldsymbol{\lambda}} \frac{1}{J(\boldsymbol{\mu})}\,\delta(\boldsymbol{x} - \boldsymbol{\mu}) \qquad J(\boldsymbol{\mu}) = \left|\det\left(\frac{\partial f_i}{\partial x_j}\right)\right|$$

Conversely, for each $\boldsymbol{\mu}$,

$$\delta(\boldsymbol{x} - \boldsymbol{\mu}) = J(\boldsymbol{\mu})\,\delta[\,\boldsymbol{f}(\boldsymbol{x})\,] \qquad \text{(g)}$$