A4.3. Stochastic Process with Prescribed Equilibrium Distribution

Let us construct **Π** for a discrete set of states assuming detailed balance.

One possibility is to set

$$\Pi_{a\,b} = \begin{cases} \frac{1}{r} & \text{if } P(a) \ge P(b) \ \ \& \ a \ne b \\ \frac{1}{r}\frac{P(a)}{P(b)} & \text{if } P(a) < P(b) \end{cases} \qquad \text{(A4.15)}$$

Assuming $P(a) > P(b)$, we have

$$\Pi_{a\,b}\,P(b) = \frac{1}{r}\,P(b)$$

$$\Pi_{b\,a}\,P(a) = \frac{1}{r}\,\frac{P(b)}{P(a)}\,P(a) = \frac{1}{r}\,P(b)$$

so that detailed balance [eq(A4.10)] is observed.

For convenience, if we re-label the states so that

$$P(a) \le P(b) \qquad \forall\ a \le b$$

we have

$$\mathbf{\Pi} = \frac{1}{r}\begin{pmatrix} \Pi_{11} & \frac{P(1)}{P(2)} & \cdots & \frac{P(1)}{P(N-1)} & \frac{P(1)}{P(N)} \\ 1 & \Pi_{22} & \cdots & \frac{P(2)}{P(N-1)} & \frac{P(2)}{P(N)} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ddots & \Pi_{N-1,N-1} & \frac{P(N-1)}{P(N)} \\ 1 & 1 & \cdots & 1 & \Pi_{N\,N} \end{pmatrix} \qquad \text{(a)}$$

The condition eq(A4.1) then gives

$$\Pi_{a\,a} + \sum_{b \ne a} \Pi_{b\,a} = 1 \qquad \text{(A4.16)}$$

which can be used to determine the diagonal elements $\Pi_{a\,a}$.

Note that $\Pi_{a\,a} > 0$ since all $\dfrac{P(a)}{P(b)} \le 1$ in eq(a).

Finally, eq(A4.16) is still valid if we scramble the state labels.

Another possibility is

$$\Pi_{a\,b} = p\,P(a)\,\theta_{a\,b} \qquad \forall\ a \ne b \qquad \text{(A4.15)}$$

with

$$\theta_{a\,b} = \theta_{b\,a} \quad \epsilon\,[\,0,\,1\,]$$

The condition eq(A4.1) then gives

$$\Pi_{a\,a} + p \sum_{b \ne a} P(a)\,\theta_{b\,a} = 1$$

Thus, p is a parameter introduced to keep

$$p \sum_{b \ne a} P(a)\,\theta_{b\,a} < 1 \qquad \forall\ a$$

so that $\Pi_{a\,a} > 0$.