5.1. Quantum Mechanics: Holomorphic Formalism

Consider the harmonic oscillator

$$H_0 = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2 \tag{5.1}$$

with $[\hat{q}, \hat{p}] = i\hbar$

Let

$$\hat{p} - i\omega\hat{q} = -i\sqrt{2\hbar\omega}\, a \qquad \hat{p} + i\omega\hat{q} = i\sqrt{2\hbar\omega}\, a^{+} \tag{5.2}$$

$$\rightarrow \quad [a, a^{+}] = \frac{1}{2\hbar\omega}[\hat{p} - i\omega\hat{q},\ \hat{p} + i\omega\hat{q}]$$

$$= \frac{i}{2\hbar}\{[\hat{p}, \hat{q}] - [\hat{q}, \hat{p}]\}$$

$$= 1 \tag{5.2a}$$

$$H_0 = \frac{1}{2}(\hat{p} + i\omega\hat{q})(\hat{p} - i\omega\hat{q}) - \frac{1}{2}i\omega[\hat{q}, \hat{p}]$$

$$= \hbar\omega\left(a^{+}a + \frac{1}{2}\right)$$

Here after, we shall assume

$$H_0 = \hbar\omega a^{+} a \tag{5.3}$$

which should be interpreted as any system with equally spaced energies

$$E_n = n\hbar\omega \qquad n = 0, 1, 2, \ ...$$

Let $\mathcal{F}$ be the complex vector space of all complex functions.

$$\left[\frac{\partial}{\partial z}, z\right] f = \frac{\partial}{\partial z}(z f) - z\frac{\partial}{\partial z} f = f \qquad \forall\, f \in \mathcal{F}$$

$$\rightarrow \quad \left[\frac{\partial}{\partial z}, z\right] = 1 \tag{5.2b}$$

The natural basis of $\mathcal{F}$ is the set of monomials $\{z^n \bar{z}^m\}$, where $\bar{z}$ is the complex conjugate of z.

An analytic function f is defined to be a function that has a Taylor series of z.
In other words,

$$f = f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{5.2c}$$

The set of all analytic functions forms a vector space $\mathcal{A}$ with the natural basis $\{z^n\}$.

Note: a holomorphic function is defined as a function that is complex differentiable (& hence infinitely differentiable) in the neighborhood of every point in its domain. Thus, it's also an analytic function.

Operating on $\mathcal{A}$, we have, by eqs(5.2a & 5.2b),

$$a \mapsto \frac{\partial}{\partial z} \qquad a^{+} \mapsto z \tag{5.4}$$

which is called the analytic (or holomorphic) representation.

Eq(5.3) then becomes

$$H_0 = \hbar\,\omega\, z\, \frac{\partial}{\partial z} \tag{5.5}$$

Using

$$z\, \frac{\partial}{\partial z}\, z^n = n\, z^n$$

we see that the Schrodinger equation

$$H_0\, \psi_n = E_n\, \psi_n$$

has solutions

$$\psi_n = c\, z^n \qquad\qquad E_n = n\,\hbar\,\omega \tag{5.5a}$$

The (imaginary time) evolution operator is

$$U_0(t) = \exp\left(-\frac{1}{\hbar} H_0\, t \right) \tag{5.6a}$$

By eq(5.5a), we have

$$U_0(t)\, z^n = \exp(-n\,\omega\, t)\, z^n$$
$$= \left(e^{-\omega t}\, z \right)^n$$

$$\rightarrow \qquad U_0(t)\, f(z) = f\left(e^{-\omega t}\, z \right) \tag{5.6}$$

Hilbert Space of Analytic Functions

By endowing to $\mathcal{A}$ an inner product

$$(g, f) = \int \frac{d\bar{z}\, d z}{2\pi i}\, e^{-z\bar{z}}\, \overline{g(z)}\, f(z) \tag{5.7}$$

we turn it into a Hilbert space $\mathcal{H}$.

The vector space $\overline{\mathcal{A}}$ spanned by the basis $\{\bar{z}^n\}$ is called the dual space of $\mathcal{A}$.

$$\left(z^m, z^n \right) = \int \frac{d\bar{z}\, d z}{2\pi i}\, e^{-z\bar{z}}\, \bar{z}^m\, z^n$$
$$= \frac{\partial^m}{\partial \bar{a}^m}\, \frac{\partial^n}{\partial a^n}\, e^{-a\bar{a}} \Big|_{a=\bar{a}=0} \qquad [\text{ Eq(1.29) of §1.3 used }]$$
$$= (-)^n\, \frac{\partial^m}{\partial \bar{a}^m} \left(\bar{a}^n\, e^{-a\bar{a}} \right) \Big|_{a=\bar{a}=0}$$

Since we need to set $\bar{a} = 0$ after the derivatives are done, the only surviving term must have no prefactor $\bar{a}$. Using

$$\frac{\partial^m}{\partial \bar{a}^m}\, \bar{a}^n = \frac{n!}{(n-m)!}\, \bar{a}^{n-m}$$

we have

$$\left(z^m, z^n \right) = n!\, \delta_{m n} \tag{5.8}$$

i.e., the basis $\{ z^n \}$ is orthogonal.

Alternatively, using [see eq(a) of §1.3]

$$\int \frac{d\bar{z}\, d z}{2 i} = \int d x\, d y = \int_0^{2\pi} d\theta \int_0^{\infty} r\, d r \qquad (z = x + i y, \quad \bar{z} = x - i y)$$

we have

$$\left(z^m, z^n\right) = \frac{1}{\pi}\int_0^{2\pi} d\,\theta \int_0^{\infty} r\,d\,r\; e^{-r^2}\, r^{m+n}\, e^{i(n-m)\theta}$$
$$= 2\,\delta_{mn}\int_0^{\infty} d\,r\; e^{-r^2}\, r^{2n+1}$$
$$= n!\,\delta_{mn}$$

as before.

For $m = n = 0$, we have

$$\int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}} = 1 \qquad (5.8a)$$

Eq(5.7) also implies the completeness relation

$$\int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}}\; |\, z\,\rangle\langle\, z\,| = I \qquad (5.8b)$$

so that

$$(g, f) = \langle g \mid f \rangle$$
$$= \int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}} \langle g \mid z\rangle\langle z \mid f\rangle$$
$$= \int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}}\, \overline{g(z)}\, f(z)$$

We'll call this the z-representation since the basis is $\{\ |\ z\,\rangle\ \}$.

By definition, a member f of $\mathcal{H}$ must have a finite norm $\|f\| = (f, f)^{1/2}$.

Using eq(5.2c), we see that $\forall\ f \in \mathcal{H}$,

$$\|f\|^2 = (f, f)$$
$$= \sum_{n,m} \overline{f_m}\, f_n \left(z^m, z^n\right)$$
$$= \sum_{n=0}^{\infty}\ |\, f_n\,|^2\ n!$$

must be finite (or square integrable). Since $f_n = \frac{1}{n!}\left.\frac{d^n f}{d\,z^n}\right|_{z=0}$, this means f is holomorphic everywhere, i.e., f is entire.

Using eq(5.8), we have

$$\int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}}\, f(z) = \sum_{n=0}^{\infty} f_n \int \frac{d\,\overline{z}\,d\,z}{2\,\pi\,i}\, e^{-z\overline{z}}\, z^n$$
$$= \sum_{n=0}^{\infty} f_n \left(z^0,\ z^n\right)$$
$$= f_0\ = f(0)$$

Thus,

$$\int \frac{d\,\overline{z}}{2\,\pi\,i}\, e^{-z\overline{z}} = \delta(z) \qquad \left[\ \text{c.f. } \delta(x) = \int \frac{d\,k}{2\,\pi}\, e^{i\,k\,x}\ \right]$$

so that

$$\int \frac{d\bar{z}\, d z}{2 \pi i} e^{-z \bar{z}} f(z) = \int d z\, \delta(z) f(z) = f(0) \qquad (5.9)$$

Operator Kernels

Since every member of $\mathcal{A}$ has a Taylor expansion, $\left\{ \langle z \mid n \rangle = \frac{z^n}{\sqrt{n!}} \right\}$ is a orthonormal basis for $\mathcal{H}$.

The completeness of the basis:

$$\sum_n \mid n \rangle \langle n \mid = I \qquad (5.9a)$$

gives

$$\sum_n \langle z \mid n \rangle \langle n \mid z' \rangle = \langle z \mid z' \rangle \qquad (5.9b)$$

so that

$$\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \frac{\bar{z}'^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{(z \bar{z}')^n}{n!} = e^{z \bar{z}'} = \langle z \mid z' \rangle$$

In fact, with

$$\langle \bar{z} \mid n \rangle = \frac{\bar{z}^n}{\sqrt{n!}}$$

the above procedure leads in general to

$$\langle \alpha \mid \beta \rangle = e^{\alpha \bar{\beta}} \quad \text{where} \quad \alpha,\ \beta = z,\ z',\ \bar{z} \text{ or } \bar{z}' \qquad (5.9c)$$

Using the completeness eq(5.8b), we have

$$\begin{aligned} f(z') &= \langle z' \mid f \rangle \\ &= \int \frac{d\bar{z}\, d z}{2 \pi i} e^{-z \bar{z}} \langle z' \mid z \rangle \langle z \mid f \rangle \\ &= \int \frac{d\bar{z}\, d z}{2 \pi i} e^{-z \bar{z}} e^{z' \bar{z}} \langle z \mid f \rangle \\ &= \int \frac{d\bar{z}\, d z}{2 \pi i} e^{-(z - z') \bar{z}} f(z) \end{aligned} \qquad (5.10)$$

which implies

$$\int \frac{d\bar{z}}{2 \pi i} e^{-(z - z') \bar{z}} = \delta(z - z') \qquad (5.10a)$$

of which eq(5.9) is a special case for $z' = 0$.

An operator is in the normal order if all the creation operators a^+'s are on the left of all the annihilation operators a's. E.g.,

$$a^{+m} a^n \;\mapsto\; z^m \frac{\partial^n}{\partial z^n}$$

is in normal order. For an arbitrary normal operator $\mathcal{N}$, we have

$$\mathcal{N}\left(z, \frac{\partial}{\partial z}\right) = \sum_{n,m} \mathcal{N}_{n m} z^m \frac{\partial^n}{\partial z^n}$$

Notation:

O = generic operator.

$\mathcal{N}$ = generic normal ordered operator.
$\mathcal{N}[O]$ = normal ordered version of O

Using

$$\frac{\partial^n}{\partial z^n} e^{z \bar{z}'} = \bar{z}'^n e^{z \bar{z}'}$$

we have

$$\mathcal{N}\left(z, \frac{\partial}{\partial z}\right) e^{z \bar{z}'} = \sum_{n,m} \mathcal{N}_{nm} z^m \frac{\partial^n}{\partial z^n} e^{z \bar{z}'} = \sum_{n,m} \mathcal{N}_{nm} z^m \bar{z}'^n e^{z \bar{z}'}$$

$$= \mathcal{N}(z, \bar{z}') e^{z \bar{z}'} \qquad (5.11)$$

$$= \mathcal{N}\left(z, \frac{\partial}{\partial z}\right) \langle z \mid z' \rangle \qquad \text{[Eq(5.9c) used]}$$

$$= \langle z \mid \mathcal{N} \mid z' \rangle \qquad (5.11p)$$

Caution: Zinn-Justin denoted eq(5.11p) as $\langle z \mid \mathcal{N} \mid \bar{z}' \rangle$ [see eq(5.11) in his text].
All subsequent eqs that bear this difference will be marked with a "p".

Consequently, all matrix elements of the form $\langle z \mid O \mid \bar{z}' \rangle$ in Zinn-Justin's text will be replaced by $\langle z \mid O \mid z' \rangle$.

Eq(5.10) gives

$$\mathcal{N}\left(z, \frac{\partial}{\partial z}\right) f(z) = \mathcal{N}\left(z, \frac{\partial}{\partial z}\right) \int \frac{d\bar{z}' dz'}{2\pi i} e^{-(z'-z)\bar{z}'} f(z')$$

Using eq(5.11), we have

$$(\mathcal{N} f)(z) = \int \frac{d\bar{z}' dz'}{2\pi i} \mathcal{N}(z, \bar{z}') e^{-(z'-z)\bar{z}'} f(z')$$

Using completeness eq(5.8b), we have

$$\langle z' \mid O_2 O_1 \mid z'' \rangle = \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \langle z' \mid O_2 \mid z \rangle \langle z \mid O_1 \mid z'' \rangle \qquad (5.12p)$$

Similarly,

$$\operatorname{tr} O = \sum_n \langle n \mid O \mid n \rangle \qquad \text{with} \quad \sum_n \mid n \rangle \langle n \mid = 1$$

$$= \sum_n \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \int \frac{d\bar{z}' dz'}{2\pi i} e^{-z'\bar{z}'} \langle n \mid z \rangle \langle z \mid O \mid z' \rangle \langle z' \mid n \rangle$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \int \frac{d\bar{z}' dz'}{2\pi i} e^{-z'\bar{z}'} \langle z \mid O \mid z' \rangle \langle z' \mid z \rangle$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \int \frac{d\bar{z}' dz'}{2\pi i} e^{-z'\bar{z}'} \langle z \mid O \mid z' \rangle e^{z'\bar{z}} \qquad \text{[Eq(5. 9c) used]}$$

$$= \int \frac{d\bar{z}' dz'}{2\pi i} e^{-z'\bar{z}'} \langle z' \mid O \mid z' \rangle \quad \text{[Eq(5.10) used]} \qquad (5.13p)$$

$$\operatorname{tr} \mathcal{N} = \int \frac{d\bar{z}\, dz}{2\pi i} \mathcal{N}(z, \bar{z}) \qquad \text{[Eq(5.11) used]} \qquad (5.13)$$

Since H_0 is normal ordered, we can use eq(5.11p) to write

$$\langle z \mid H_0 \mid z' \rangle = H_0(z, \bar{z}') e^{z\bar{z}'}$$
$$= \hbar\,\omega\, z\, \bar{z}' e^{z\bar{z}'} \qquad (5.14p)$$

Bearing in mind that U_0 is not normal ordered, we have

$$\langle z \mid U_0(t) \mid z' \rangle = \left\langle z \,\middle|\, \exp\left(-\frac{1}{\hbar} H_0\, t\right) \middle|\, z' \right\rangle \qquad \text{[Eq(5.6a) used]}$$
$$= \exp\left(-\frac{1}{\hbar} H_0\, t\right) e^{z\bar{z}'}$$
$$= \sum_{n=0}^{\infty} \frac{(-\omega\, t)^n}{n!} \left(z \frac{\partial}{\partial z}\right)^n e^{z\bar{z}'}$$

Let

$$P = z \frac{\partial}{\partial z} \qquad \rho = z\,\bar{z}'$$

then

$$P\,\rho^n = n\,\rho^n \qquad P^m\,\rho^n = n^m\,\rho^n$$

Hence

$$\langle z \mid U_0(t) \mid z' \rangle = \sum_{n=0}^{\infty} \frac{(-\omega\, t)^n}{n!} P^n \sum_{m=0}^{\infty} \frac{\rho^m}{m!}$$
$$= \sum_{n,m=0}^{\infty} \frac{(-\omega\, t)^n}{n!\, m!} m^n \rho^m$$
$$= \sum_{m=0}^{\infty} \frac{e^{-m\,\omega\, t}}{m!} \rho^m$$
$$= \exp\left(e^{-\omega\, t}\, \rho\right)$$
$$= \exp\left(e^{-\omega\, t}\, z\,\bar{z}'\right) \qquad (5.14p)$$

The corresponding partition function is

$$\mathcal{Z}_0(\beta) = \mathrm{tr}\, U_0(\hbar\,\beta)$$
$$= \int \frac{d\bar{z}\, d z}{2\pi i} e^{-z\bar{z}} \langle z \mid U_0 \mid z \rangle \qquad \text{[Eq(5.13p) used]}$$
$$= \int \frac{d\bar{z}\, d z}{2\pi i} \exp\left[-\left(1 - e^{-\beta\,\hbar\,\omega}\right) z\,\bar{z}\right] \qquad \text{[Eq(5.14p) used]}$$

Using

$$\frac{d\bar{z}\, d z}{2 i} = d x\, d y = r\, d r\, d\theta \qquad \& \qquad z\,\bar{z} = r^2$$

we have

$$\mathcal{Z}_0(\beta) = \frac{1}{\pi} \int_0^{\infty} r\, d r \int_0^{2\pi} d\theta\, \exp\left[-\left(1 - e^{-\beta\,\hbar\,\omega}\right) r^2\right]$$
$$= \int_0^{\infty} d x\, \exp\left[-\left(1 - e^{-\beta\,\hbar\,\omega}\right) x\right]$$
$$= \frac{1}{1 - e^{-\beta\,\hbar\,\omega}} \qquad (5.15)$$

Remarks

(i) From eqs(5.11 & 5.11p), we have

$$\langle z \mid O \mid z' \rangle = O\left(z, \frac{\partial}{\partial z}\right) e^{z \bar{z}'} \qquad \text{(a)}$$

$$\rightarrow \quad \overline{\langle z \mid O \mid z' \rangle} = \langle z' \mid O^{+} \mid z \rangle$$

$$= O^{+}\left(z', \frac{\partial}{\partial z'}\right) e^{z' \bar{z}} \qquad \text{[Eq(a) used]}$$

$$= \overline{O\left(z, \frac{\partial}{\partial z}\right)} e^{\bar{z} z'} \qquad \text{[Eq(a) used]}$$

$$\therefore \quad O^{+}\left(z', \frac{\partial}{\partial z'}\right) = \overline{O\left(z, \frac{\partial}{\partial z}\right)}$$

For a normal operator, $\frac{\partial}{\partial z}$ in eq(a) can be replaced by $\bar{z}$, so that

$$\mathcal{N}^{+}(z', \bar{z}) = \overline{\mathcal{N}(z, \bar{z}')}$$

For a hermitian operator

$$O\left(z', \frac{\partial}{\partial z'}\right) = \overline{O\left(z, \frac{\partial}{\partial z}\right)}$$

In particular, setting $z' = z$ to obtain the diagonal elements, we have

$$O\left(z, \frac{\partial}{\partial z}\right) = \overline{O\left(z, \frac{\partial}{\partial z}\right)}$$

Therefore, if $\mathcal{N}$ is hermitian, then

$$\mathcal{N}(z', \bar{z}) = \overline{\mathcal{N}(z, \bar{z}')}$$

& in particular, $\mathcal{N}(z, \bar{z})$ must be real.

Since H_0 is normal ordered & using eq(5.14p), we have

$$H_0(z, \bar{z}') = \hbar \omega z \bar{z}'$$

$$\rightarrow \quad H_0(z', \bar{z}) = \hbar \omega z' \bar{z}$$

$$\overline{H_0(z, \bar{z}')} = \hbar \omega \bar{z} z' = H_0(z', \bar{z})$$

$\therefore$ H_0 is hermitian.

Using eq(5.14p),we see that U_0 is also hermitian.

(ii) Using [see eq(5.10d)]

$$I = \int \frac{d\bar{z}\, dz}{2 \pi i} e^{-z \bar{z}} \mid z \rangle \langle z \mid \qquad \& \qquad \sum_n \mid n \rangle \langle n \mid = I$$

we have

$$O = \sum_{m,n} \mid m \rangle \langle m \mid O \mid n \rangle \langle n \mid$$

$$= \sum_{m,n} \int \frac{d\bar{z}\, dz}{2 \pi i} e^{-z \bar{z}} \int \frac{d\bar{z}'\, dz'}{2 \pi i} e^{-z' \bar{z}'} \mid z \rangle \langle z \mid m \rangle \langle m \mid O \mid n \rangle \langle n \mid z' \rangle \langle z' \mid$$

$$= \sum_{m,n} \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \int \frac{d\bar{z'}\, dz'}{2\pi i} e^{-z'\bar{z'}} \mid z\rangle \frac{z^m}{\sqrt{m!}} O_{mn} \frac{\bar{z}'^n}{\sqrt{n!}} \langle z' \mid$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} \int \frac{d\bar{z'}\, dz'}{2\pi i} e^{-z\bar{z}-z'\bar{z'}} \mid z\rangle\langle z' \mid \sum_{m,n} O_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}'^n}{\sqrt{n!}}$$

where

$$O_{mn} = \langle m \mid O \mid n\rangle$$

Thus, the operator O is characterized by the kernel

$$\sum_{m,n} O_{mn} \frac{z^m}{\sqrt{m!}} \frac{\bar{z}'^n}{\sqrt{n!}}$$

Remark (iii) is irrelevant to our version of derivation.