5.2. Holomorphic Path Integral

We wish to derive the path integral representation of $\langle z \mid U(t, t') \mid \bar{z} \rangle$ where

$$\hbar \frac{\partial U(t, t')}{\partial t} = -H(t)\, U(t, t') \qquad\qquad U(t', t') = 1 \tag{5.16}$$

5.2.1. The Harmonic Oscillator

Cuation: Despite the section title, we're actually dealing with a system that can assume only energies

$$E_m = m \hbar \omega \qquad\qquad m = 0, 1, 2, \ldots$$

which is interpreted as a state of energy $\hbar\omega$ being occupied by m bosons.

Using eq(5.14p), we have

$$\langle z \mid U_0(\varepsilon) \mid z' \rangle = \exp\left[z\, \bar{z}' (1 - \omega\varepsilon) + O(\varepsilon^2) \right] \tag{5.17p}$$

$$\rightarrow \qquad \langle z_{k+1} \mid U_0(\varepsilon) \mid z_k \rangle = \exp\left[z_{k+1}\, \bar{z}_k (1 - \omega\varepsilon) + O(\varepsilon^2) \right]$$

Using the completeness eq(5.8b), the group property eq(5.12) now takes the form

$$\langle z'' \mid U_0(t'', t') \mid z' \rangle = \langle z'' \mid U_0(t'', t)\, U_0(t, t') \mid z' \rangle$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \langle z'' \mid U_0(t'', t) \mid z \rangle \langle z \mid U_0(t, t') \mid z' \rangle$$

Note: Analogous expression cannot be obtained for $\langle z'' \mid U_0(t'', t') \mid \bar{z}' \rangle$ partly because one cannot set

$$\int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \mid \bar{z} \rangle \langle \bar{z} \mid = 1$$

since it implies

$$(g, f) = \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}} \langle g \mid \bar{z} \rangle \langle \bar{z} \mid f \rangle$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}}\, \overline{g(\bar{z})}\, f(\bar{z})$$

$$= \int \frac{d\bar{z}\, dz}{2\pi i} e^{-z\bar{z}}\, g(z)\, \overline{f(z)}$$

in disagreement with the definition eq(5.7).

Setting

$$\varepsilon = \frac{t'' - t'}{n} \qquad\qquad t_k = t' + k\varepsilon \qquad\qquad n = \text{odd}$$

$$z_k = z(t_k) \qquad\qquad \bar{z}_k = \bar{z}(t_k)$$

$$z_0 = z' \qquad\qquad z_n = z'' \tag{5.20}$$

we have

$$\langle z'' \mid U_0(t'', t') \mid z' \rangle = \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{d\bar{z}_k\, dz_k}{2\pi i} e^{-z_k \bar{z}_k} \langle z_{k+1} \mid U_0(t_{k+1}, t_k) \mid z_k \rangle$$

$$\times \langle z_1 \mid U_0(t_1, t_0) \mid z_0 \rangle$$

$$= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{d\bar{z}_k\, dz_k}{2\pi i} e^{-S_\varepsilon(z,\bar{z})} \tag{5.18p}$$

where

$$S_\varepsilon(z,\bar{z}) = \sum_{k=1}^{n-1} z_k\,\bar{z}_k - \sum_{k=0}^{n-1} z_{k+1}\,\bar{z}_k\,(1-\omega\,\varepsilon)$$
$$= \sum_{k=1}^{n-1} (z_k - z_{k+1})\,\bar{z}_k - z_0\,\bar{z}_1 + \varepsilon\,\omega \sum_{k=0}^{n-1} z_{k+1}\,\bar{z}_k \qquad (5.19)$$
$$= -z_0\,\bar{z}_1 + \varepsilon\left(-\sum_{k=1}^{n-1}\left(\frac{z_{k+1}-z_k}{\varepsilon}\right)\bar{z}_k + \omega \sum_{k=0}^{n-1} z_{k+1}\,\bar{z}_k\right)$$

As $\varepsilon \to 0$ & $n \to \infty$, eq(5.18) becomes a path integral

$$\langle z'' \mid U_0(t'', t') \mid z' \rangle = \int [\,d\,\bar{z}(t)\,d\,z(t)\,]\,e^{-S_0(z,\bar{z})} \qquad (5.21p)$$

with

$$[\,d\,\bar{z}(t)\,d\,z(t)\,] = \prod_{k=1}^{n-1} \frac{d\,\bar{z}_k\,d\,z_k}{2\,\pi\,i}$$
$$S_0(z,\bar{z}) = -z(t')\,\bar{z}(t') + \int_{t'}^{t''} d\,t\;\left[-\dot{z}(t) + \omega\,z(t)\right]\bar{z}(t) \qquad (5.21a)$$

& $\qquad z(t'') = z'' \qquad\qquad z(t') = z' \qquad (5.21b)$

The paths are “phase space” trajectories $\{\,z(t),\,\bar{z}(t)\,\}$.

Unlike the real-time evolution case, no symmetry of S_0 with respect to the initial & final times was found.

The Partition Function

Using eq(5.13) on eqs(5.18-9), we have

$$\mathcal{Z}_0(\beta) = \mathrm{tr}\,U_0(\,\hbar\beta/2\,,\,-\hbar\beta/2\,)$$
$$= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} \frac{d\,\bar{z}_k\,d\,z_k}{2\,\pi\,i}\,e^{-S_\varepsilon(z,\bar{z})} \qquad (5.22)$$

where

$$S_\varepsilon(z,\bar{z}) = \sum_{k=1}^{n} z_k\,\bar{z}_k - \sum_{k=0}^{n-1} z_{k+1}\,\bar{z}_k\,(1-\omega\,\varepsilon)$$
$$= \sum_{k=1}^{n} z_k\,\bar{z}_k - \sum_{k=1}^{n} z_k\,\bar{z}_{k-1}\,(1-\omega\,\varepsilon)$$
$$= \varepsilon \sum_{k=1}^{n}\left(z_k\,\frac{\bar{z}_k - \bar{z}_{k-1}}{\varepsilon} + \omega\,z_k\,\bar{z}_{k-1}\right) \qquad (5.23a)$$

In the continuum limit with periodic boundary conditions

$$z(\hbar\beta/2) = z(-\hbar\beta/2) \qquad \dot{z}\,(\hbar\beta/2) = \dot{z}(-\hbar\beta/2)$$
$$\bar{z}\,(\hbar\beta/2) = \bar{z}(-\hbar\beta/2) \qquad \dot{\bar{z}}\,(\hbar\beta/2) = \dot{\bar{z}}(-\hbar\beta/2) \qquad (5.24a)$$

we have

$$\mathcal{Z}_0(\beta) = \int [\,d\,\bar{z}(t)\,d\,z(t)\,]\;e^{-S_0(z,\bar{z})} \qquad (5.24)$$
$$S_0\,(z,\bar{z}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\left(z\,\dot{\bar{z}} + \omega\,z\,\bar{z}\right)$$
$$= \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\left(-\dot{z}\,\bar{z} + \omega\,z\,\bar{z}\right) \qquad \text{(Integration by part done.)}$$

$$= \int_{-\beta/2}^{\beta/2} d\tau \left(-\frac{dz}{d\tau} \bar{z} + \hbar \omega z \bar{z} \right) \qquad \left(\tau = \frac{t}{\hbar} \right) \tag{5.25}$$

Generating Functional

Warning: From here-on, all t's in Zinn-Justin's text are actually $\tau = \frac{t}{\hbar}$ [see eq(5.25)].

This means $\hbar$'s in our equations will appear at different locations in Zinn-Justin's.
If this leads to different definitions of a quantity, a "t" will be attached to the equation's label.

Let

$$\begin{aligned} H(t) &= H_0 - \hbar \bar{b}(t) a^+(t) - \hbar b(t) a(t) \\ &= \hbar \left[\omega a^+(t) a(t) - \bar{b}(t) a^+(t) - b(t) a(t) \right] \end{aligned} \tag{5.26t}$$

The solution to eq(5.16) is given by

$$\begin{aligned} U_G(t+\varepsilon, t) &= U_G(t,t) - \frac{1}{\hbar} \int_t^{t+\varepsilon} dt' H(t') U_G(t', t) \qquad U_G(t,t) = 1 \\ &= 1 - \frac{\varepsilon}{\hbar} H(t) + O(\varepsilon^2) \\ &= 1 - \varepsilon \left[\omega a^+(t) a(t) - \bar{b}(t) a^+(t) - b(t) a(t) \right] + O(\varepsilon^2) \end{aligned}$$

Using eqs(5.4 & 5.9c), we have

$$\begin{aligned} \langle z \mid U_G(t+\varepsilon, t) \mid z' \rangle &= \left\langle z \,\middle|\, 1 - \frac{\varepsilon}{\hbar} H(t) \,\middle|\, z' \right\rangle + O(\varepsilon^2) \\ &= \left\{ 1 - \varepsilon \left[\omega z \frac{\partial}{\partial z} - \bar{b}(t) z - b(t) \frac{\partial}{\partial z} \right] \right\} e^{z \bar{z}'} + O(\varepsilon^2) \\ &= \left\{ 1 - \varepsilon \left[\omega z \bar{z}' - \bar{b}(t) z - b(t) \bar{z}' \right] \right\} e^{z \bar{z}'} + O(\varepsilon^2) \\ &= \exp\left\{ z \bar{z}' - \varepsilon \left[\omega z \bar{z}' - \bar{b}(t) z - b(t) \bar{z}' \right] \right\} + O(\varepsilon^2) \end{aligned}$$

$$\rightarrow \qquad \langle z_{k+1} \mid U_G(t_{k+1}, t_k) \mid z_k \rangle = \exp\left\{ z_{k+1} \bar{z}_k - \varepsilon \left[\omega z_{k+1} \bar{z}_k - \bar{b}_k z_{k+1} - b_k \bar{z}_k \right] \right\} + O(\varepsilon^2)$$

Since the b terms are local in z, one simply adds them to the action of eq(5.21a) so that

$$S_G(z, \bar{z}) = -z(t') \bar{z}(t') + \int_{t'}^{t''} dt \left\{ \left[-\dot{z}(t) + \omega z(t) \right] \bar{z}(t) - \bar{b}(t) z(t) - b(t) \bar{z}(t) \right\} \tag{5.27}$$

Similarly, eqs(5.54-55) become

$$\begin{aligned} \mathcal{Z}_G(\beta) &= \operatorname{tr} U_G(\beta\hbar/2, -\beta\hbar/2) \\ &= \int [d\bar{z}(t)\, dz(t)]\, e^{-S_G(z,\bar{z})} \end{aligned} \tag{5.28}$$

$$S_G(z, \bar{z}) = \int_{-\beta\hbar/2}^{\beta\hbar/2} dt \left\{ \left[-\dot{z}(t) + \omega z(t) \right] \bar{z}(t) - \bar{b}(t) z(t) - b(t) \bar{z}(t) \right\} \tag{5.28a}$$

with the periodic boundary conditions eq(5.24a).

The classical equation of motion (with $\frac{\partial}{\partial z} \propto$ momentum) is given by

$$\begin{aligned} \dot{\bar{z}}_c &= -\frac{1}{\hbar} \left[\frac{\partial}{\partial z}, H \right]_{z = z_c} \qquad [\, t = i \times \text{"real time"} \,] \\ &= -\frac{1}{\hbar} \left[\frac{\partial}{\partial z}, \hbar \left(\omega z \frac{\partial}{\partial z} - \bar{b} z - b \frac{\partial}{\partial z} \right) \right]_{z = z_c} \\ &= -\omega \bar{z}_c + \bar{b} \end{aligned} \tag{a}$$

Note: $\dot{z} = -\frac{1}{\hbar}[z, H]$ cannot be used as the classical equation of motion since $\dot{z}$ is just a "velocity".

Alternatively, using

$$S = \int dt\, L$$

the Euler-Lagrange equation for eq(5.28a) is obtained as follows

$$\frac{\partial L}{\partial z} = \omega \bar{z} - \bar{b} \qquad\qquad \frac{\partial L}{\partial \dot{z}} = -\bar{z}$$

→ $\omega \bar{z} - \bar{b} + \dot{\bar{z}} = 0$

Setting $z = z_c$, we recover eq(a).

The corresponding green's function is given by

$$\dot{\Delta}(t) + \omega \Delta(t) = \delta(t) \qquad (5.31t)$$

Integrating eq(5.31a) for $t \in [-\varepsilon, \varepsilon]$ gives the discontinuity

$$\Delta(\varepsilon) - \Delta(-\varepsilon) = 1 \qquad \text{as } \varepsilon \to 0$$

Since the homogenous solution is $\Delta(t) = c\, e^{-\omega t}$, we have

$$\Delta(t) = \begin{cases} c\, e^{-\omega t} & \text{for } t > 0 \\ c'\, e^{-\omega t} & \text{for } t < 0 \end{cases}$$

where the discontinuity at $z = 0$ & periodic B.C. [eq(5.24a)] give

$$c - c' = 1 \qquad \& \qquad c\, e^{-\beta \hbar \omega/2} = c'\, e^{\beta \hbar \omega/2}$$

→ $$c = \frac{1}{1 - e^{-\beta \hbar \omega}} = \frac{e^{\beta \hbar \omega/2}}{2 \sinh(\beta \hbar \omega/2)}$$

$$c' = \frac{e^{-\beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} = \frac{e^{-\beta \hbar \omega/2}}{2 \sinh(\beta \hbar \omega/2)}$$

∴ $$\Delta(t) = \frac{e^{-\omega t}}{1 - e^{-\beta \hbar \omega}} \begin{cases} 1 & \text{for } t > 0 \\ e^{-\beta \hbar \omega} & \text{for } t < 0 \end{cases} \qquad (5.30a)$$

$$= \frac{e^{-\omega t}}{2\left(1 - e^{-\beta \hbar \omega}\right)} \left[\epsilon(t) \left(1 - e^{-\beta \hbar \omega}\right) + 1 + e^{-\beta \hbar \omega} \right]$$

where

$$\epsilon(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

Since $\epsilon(t)$ is a step function, its derivative is a delta function, i.e.,

$$\dot{\epsilon}(t) = c\, \delta(t)$$

Integrating for $t \in [-\varepsilon, \varepsilon]$ gives

$$\epsilon(\varepsilon) - \epsilon(-\varepsilon) = 2 = c$$

∴ $\dot{\epsilon}(t) = 2\, \delta(t)$

→ $$\Delta(t) = \frac{e^{-\omega t}}{2} \left[\epsilon(t) + \frac{1 + e^{-\beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} \right]$$

$$= \frac{e^{-\omega t}}{2} \left[\epsilon(t) + \frac{1}{\tanh(\beta \hbar \omega/2)} \right] \qquad (5.30)$$

In the saddle-point approximation, we replace $\bar{z}$ with $\bar{z}_c$ so that eq(5.28a) becomes

$$S_G(z, \bar{z}_c) = \int_{-\beta\hbar/2}^{\beta\hbar/2} dt \left\{ \left(-\dot{z} + \omega z \right) \bar{z}_c - \bar{b} z - b \bar{z}_c \right\}$$

Using the periodic B.C. eq(5.24a) to integrate by part the 1st term, we have

$$S_G(z, \bar{z}_c) = \int_{-\beta\hbar/2}^{\beta\hbar/2} dt \left\{ \left(\dot{\bar{z}}_c + \omega \bar{z}_c - \bar{b} \right) z - b \bar{z}_c \right\}$$
$$= -\int_{-\beta\hbar/2}^{\beta\hbar/2} dt \;\; b \bar{z}_c$$
$$= -\int_{-\beta\hbar/2}^{\beta\hbar/2} dt \int_{-\hbar\beta/2}^{\hbar\beta/2} dt\, du\; b(t)\, \Delta(t-u)\, \bar{b}(u)$$

Hence,

$$\mathcal{Z}_G(\beta) = \mathcal{Z}_0(\beta) \int [d\bar{r}(t)\, dr(t)]\; e^{-S_G(z,\bar{z})}$$
$$= \mathcal{Z}_0(\beta) \exp\left[\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\, du\;\; b(t)\, \Delta(t-u)\, \bar{b}(u) \right] \qquad (5.29)$$

so that [see eq(2.48b) of §2.5.2]

$$\langle \bar{z}(t_2)\, z(t_1) \rangle_0 = \frac{1}{\mathcal{Z}_0(\beta)} \frac{\delta^2}{\delta \bar{b}(t_2)\, \delta b(t_1)} \mathcal{Z}_G(b, \beta) \Big|_{b=\bar{b}=0}$$
$$= \Delta(t_2 - t_1) \qquad (5.32)$$

Verification

From eqs(5.24-5), we have

$$\frac{\partial S_0}{\partial \omega} = \int_{-\hbar\beta/2}^{\hbar\beta/2} dt\; z\bar{z}$$
$$\frac{\partial}{\partial \omega} \ln \mathcal{Z}_0(\beta) = \frac{1}{\mathcal{Z}_0(\beta)} \frac{\partial}{\partial \omega} \mathcal{Z}_0(\beta)$$
$$= -\frac{1}{\mathcal{Z}_0(\beta)} \int [d\bar{z}(t)\, dz(t)]\, \frac{\partial S_0}{\partial \omega}\, e^{-S_0(z,\bar{z})}$$
$$= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\, \frac{1}{\mathcal{Z}_0(\beta)} \int [d\bar{z}(t')\, dz(t')]\, z(t)\, \bar{z}(t)\, e^{-S_0(z,\bar{z})}$$
$$= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \langle\, z(t)\, \bar{z}(t)\, \rangle_0$$
$$= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\, \Delta(0) \qquad \text{[Eq(5.32) used]}$$
$$= -\hbar\beta\, \Delta(0) \qquad (5.32a)$$

From eq(5.30b), we have

$$\Delta(0) = \frac{1}{2} \left[\epsilon(0) + \frac{1}{\tanh(\beta\hbar\omega/2)} \right]$$

Eq(5.32a) then integrates to give

$$\ln \mathcal{Z}_0(\beta) = -\frac{1}{2} \hbar\beta\omega\, \epsilon(0) - \frac{\hbar\beta}{2} \int \frac{d\omega}{\tanh(\beta\hbar\omega/2)} + C$$
$$= -\frac{1}{2} \hbar\beta\omega\, \epsilon(0) - \ln\sinh(\beta\hbar\omega/2) + C$$

As $\beta \to \infty$, we have

$$\ln \mathcal{Z}_0(\beta) = -\beta E_0 = 0$$
$$= -\frac{1}{2} \hbar\beta\omega\, \epsilon(0) - \frac{1}{2} \hbar\beta\omega + \ln 2 + C \qquad \left(\ln\sinh x \to \ln \frac{e^x}{2} = x - \ln 2 \right)$$

$$\therefore \qquad C = \frac{1}{2}\hbar\beta\omega\,\epsilon(0) + \frac{1}{2}\hbar\beta\omega - \ln 2$$

$$\ln \mathcal{Z}_0(\beta) = \frac{1}{2}\hbar\beta\omega - \ln\left[2\sinh(\beta\hbar\omega/2)\right]$$

$$= -\ln\left[2\,e^{-\beta\hbar\omega/2}\sinh(\beta\hbar\omega/2)\right]$$

$$\rightarrow \qquad \mathcal{Z}_0(\beta) = \frac{e^{\beta\hbar\omega/2}}{2\sinh(\beta\hbar\omega/2)}$$

$$= \frac{1}{1-e^{-\beta\hbar\omega}} \tag{5.33}$$

Unlike Zinn-Justin, we encounter no "$\epsilon(0)$" issues.

5.2.2. General Hamiltonian: One Degree of Freedom

Eqs(5.21p & 5.21a) can obviously be generalized to

$$\langle z''\,|\,U_0(t'',t')\,|\,z'\rangle = \int [d\bar{z}(t)\,dz(t)]\,e^{-S(z,\bar{z})} \tag{5.34p}$$

with

$$S(z,\bar{z}) = -z(t')\,\bar{z}(t') + \int_{t'}^{t''} dt\left(-\dot{z}(t)\,\bar{z}(t) + \frac{1}{\hbar}h[z(t),\bar{z}(t)]\right) \tag{5.35}$$

$$\& \qquad z(t'') = z'' \qquad\qquad z(t') = z' \tag{5.35a}$$

where $h(z,\bar{z})$ is obtained by replacing $\frac{\partial}{\partial z}$ with $\bar{z}$ in the hamiltonian $h\left(z, \frac{\partial}{\partial z}\right)$.

Real Parametrization of Phase Space

If we use $\bar{z}$ to represent $\frac{\partial}{\partial z}$, then for a harmonic oscillator [see eqs(5.1 & 5.4)]

$$p - i\,\omega\,q = -i\sqrt{2\hbar\omega}\;a = -i\sqrt{2\hbar\omega}\;\bar{z} \tag{5.36}$$

$$p + i\,\omega\,q = i\sqrt{2\hbar\omega}\;a^{+} = i\sqrt{2\hbar\omega}\;z$$

where (p, q) & $(z,\bar{z})$ are now merely variables that parametrize the phase space.

Inverting eq(5.36), we have

$$p = i\sqrt{\frac{\hbar\omega}{2}}\,(z-\bar{z}) \qquad\qquad q = \sqrt{\frac{\hbar\omega}{2}}\,(z+\bar{z}) \tag{5.36a}$$

$$\rightarrow \qquad h(p,q) = h\left[i\sqrt{\frac{\hbar\omega}{2}}\,(z-\bar{z}),\ \sqrt{\frac{\hbar\omega}{2}}\,(z+\bar{z})\right]$$

$$= h(z,\bar{z})$$

Hence, the phase space path integrals studied in Chapter 3 can be directly converted to holomorphic path integrals. Unfortunately, both formulism have operator ordering problems.

Note that by treating q & p as generalized coordinates & momentum, repectively, we can apply the above to any system with energies $E_m = m\hbar\omega$.

Partition Function

Eqs(5.24 & 5.25) generalize to

$$\mathcal{Z}(\beta) = \int [\, d\,\overline{z}(t)\, d\, z(t)\,]\ e^{-S(z,\overline{z})}$$

$$S(z, \overline{z}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \left(-\dot{z}\,\overline{z} + \frac{1}{\hbar}\, h(z, \overline{z}) \right)$$

with periodic boundary conditions eq(5.24a).

Writing

$$h = \hbar\,\omega\; z\,\overline{z} + h_I(z, \overline{z})$$

one can evaluate $\mathcal{Z}(\beta)$ by perturbation. Using Wick's theorem, one needs only evaluate various 2-point correlation functions.

Remark

If [see Chapter 2]

$$H = \frac{1}{2}\,\hat{p}^2 + \frac{1}{2}\,\omega^2\,\hat{q}^2 + V_I(\hat{q})$$

then perturbation requires only the calculation of 2-point functions of $q(t) \propto z + \overline{z}$. These can be generated from an action S_G [see eq(5.27)] with $b = \overline{b}$ so that the source term reduces to

$$-b(t)[\, z(t) + \overline{z}(t)\,]$$

Using

$$\int d\,t\;\; d\,u\;\; b(t)\,\Delta(t-u)\,b(u) = \frac{1}{2}\int d\,t\;\; d\,u\,[\; b(t)\,\Delta(t-u)\,b(u) + \; b(u)\,\Delta(u-t)\,b(t)\,]$$

$$= \frac{1}{2}\int d\,t\;\; d\,u\;\; b(t)\,\tilde{\Delta}(t-u)\,b(u)$$

where

$$\tilde{\Delta}(t-u) = \Delta(t-u) + \Delta(u-t)\;\; = \tilde{\Delta}(u-t) \qquad (5.38a)$$

eq(5.29) can be symmetrized as

$$\mathcal{Z}_G(\beta) = \mathcal{Z}_0(\beta)\,\exp\!\left[\, \frac{1}{2}\int d\,t\;\; d\,u\;\;\; b(t)\,\tilde{\Delta}\,(t-u)\,b(u)\,\right]$$

Using eq(5.33), we have

$$\mathcal{Z}_G(\beta) = \frac{1}{2\,\sinh\,(\beta\,\hbar\,\omega\,/2\,)}\,\exp\!\left[\, \frac{1}{2}\int d\,t\;\; d\,u\;\;\; b(t)\,\tilde{\Delta}\,(t-u)\,b(u)\,\right] \qquad (5.37a)$$

Using eq(5.30a), eq(5.38a) becomes

$$\begin{aligned}
\tilde{\Delta}\,(t) &= \;\Delta(t) + \Delta(-t) \\
&= \frac{1}{1 - e^{-\beta\hbar\omega}} \begin{cases} e^{-\omega t} + e^{\omega t - \beta\hbar\omega} & \text{for } t > 0\ \ \&\ -t < 0 \\ e^{-\omega t - \beta\hbar\omega} + e^{\omega t} & \text{for } t < 0\ \ \&\ -t > 0 \end{cases} \\
&= \frac{1}{1 - e^{-\beta\hbar\omega}} \begin{cases} e^{-\omega\,|t|} + e^{\omega\,|t| - \beta\hbar\omega} & \text{for } t > 0 \\ e^{\omega\,|t| - \beta\hbar\omega} + e^{-\omega\,|t|} & \text{for } t < 0 \end{cases} \\
&= \frac{e^{-\omega\,|t|} + e^{\omega\,|t| - \beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \\
&= \frac{e^{\,\beta\hbar\omega/2 - \omega\,|t|} + e^{\,\omega\,|t| - \beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}} \\
&= \frac{\cosh[\,\omega\,(\;\beta\,\hbar/2 - \mid t \mid\,)\,]}{\sinh(\,\beta\,\hbar\,\omega\,/2\,)}
\end{aligned}$$

(5.38b)

In order to compare with eq(2.45) of §2.5, we rescale b & set

$$b = \tilde{b}\sqrt{\frac{\hbar}{2\,\omega}}$$

so that eq(5.37a) becomes

$$\mathcal{Z}_G(\beta) = \frac{1}{2\sinh(\beta\,\hbar\,\omega\,/2)}\exp\Big[\frac{1}{2}\int d\,t\;d\,u\;\;\tilde{b}\,(t)\,\tilde{\tilde{\Delta}}(t-u)\,\tilde{b}\,(u)\Big] \qquad (5.37)$$

where

$$\tilde{\tilde{\Delta}}(t) = \hbar\,\frac{\cosh[\,\omega\,(\,\beta\,\hbar/2 - |\,t\,|\,)]}{2\,\omega\sinh(\,\beta\,\hbar\,\omega\,/2\,)}$$

(5.38)
in agreement with eq(2.45) in §2.5.

5.2.3 Several Degrees of Freedom: Many-Body Interpretation

For N non-interacting oscillators, eqs(5.5 & 5.5a) generalize to

$$H_0 = \sum_{i=1}^{N}\hbar\,\omega_i\,z_i\,\frac{\partial}{\partial\,z_i} \qquad \text{with} \qquad E_n = \hbar\sum_i n_i\,\omega_i$$

Following the usual 2nd quantization scheme, this can be interpreted as representing $\sum_i n_i$ independent particles occupying N different states of energies $\hbar\,\omega_i$ according to the occupation number $n = (\,n_1,\;\ldots,\,n_N\,)$.

A vector $\psi(\mathbf{z}) = \psi\,(z_1,\;\ldots,\,z_N)$ in the holomorphic Hilbert space now has the Taylor expansion

$$\psi\,(\mathbf{z}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i_1,\ldots,i_n}\psi_{i_1\ldots i_n}\,z_{i_1}\,\ldots\,z_{i_n} \qquad \psi_{i_1\ldots i_n} = \left.\frac{\partial^n\,\psi}{\partial\,z_{i_1}\ldots\partial\,z_{i_n}}\right|_0$$

Since the z_i's commute among themselves, the particles are bosons.

Eqs(5.24-5) are easily generalized to read

$$\mathcal{Z}_0(\beta) = \int[\,d\,\overline{\mathbf{z}}(t)\,d\,\mathbf{z}(t)\,]\;e^{-S_0(\mathbf{z},\overline{\mathbf{z}})} \qquad (5.40)$$

$$S_0(\mathbf{z},\,\overline{\mathbf{z}}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\sum_{i=1}^{N}\left(-\dot{z}_i\,\overline{z}_i + \omega_i\,z_i\,\overline{z}_i\right) \qquad (5.41t)$$