5.3. Path Integrals with Fermions

See §§1.4-7 for the basics of Grassmann algebras.

5.3.1. Fermions and Complex Vector Spaces

Let $\mathbb{C}$ be a Grassmann (or Clifford) algebra with 2 sets of generators

$$\{\theta_i, \overline{\theta}_i\}, \ i = 1, \ \ldots, n$$

The hermitian conjugation is defined as

$$(A B)^+ = B^+ A^+ \quad \forall \ A, B \in \mathbb{C}$$

and

$$\theta_i^+ = \overline{\theta}_i \qquad \overline{\theta}_i^+ = \theta_i \tag{5.42}$$

A quadratic form

$$\boldsymbol{\theta}^+ \boldsymbol{M} \boldsymbol{\theta} = \sum_{i,j=1}^{n} \overline{\theta}_i M_{ij} \theta_j \qquad \boldsymbol{\theta} = (\theta_1, \ \ldots, \ \theta_n)^T$$

is said to be invariant if

$$\left(\boldsymbol{\theta}^+ \boldsymbol{M} \boldsymbol{\theta}\right)^+ = \boldsymbol{\theta}^+ \boldsymbol{M}^+ \boldsymbol{\theta} = \sum_{i,j=1}^{n} \overline{\theta}_i M_{ij}^+ \theta_i$$

$$= \boldsymbol{\theta}^+ \boldsymbol{M} \boldsymbol{\theta} = \sum_{i,j=1}^{n} \overline{\theta}_i M_{ij} \theta_i$$

i.e., $\boldsymbol{M}^+ = \boldsymbol{M}$ or $\overline{M}_{ji} = M_{ij}$

For example, $d\theta_i \, d\overline{\theta}_i$ is invariant since, by eq(5.42),

$$\left(d\theta_i \, d\overline{\theta}_i\right)^+ = d\theta_i \, d\overline{\theta}_i$$

Analytic Grassmann Functions

An analytic Grassmann function f is a function of $\boldsymbol{\theta}$ only, i.e.,

$$\frac{\partial f}{\partial \overline{\theta}_i} = 0 \qquad \forall \ i = 1, \ \ldots, n$$

The set $\mathfrak{U}$ of all analytic Grassmann functions forms a sub-algebra.

For the case of $n = 1$, the set of all analytic $f(\theta)$ forms a 2-D complex vector space with basis $\{1, \ \theta\}$ (see §1.4). Physically, 1 & θ represent the state being empty or occupied by a fermion, respectively.

Scalar Product of Analytic Functions

The counterpart of eq(5.7) is

$$(f, g) = \int \prod_{i=1}^{n} d\theta_i \, d\overline{\theta}_i \exp\left(\sum_{j=1}^{n} \overline{\theta}_j \theta_j\right) \overline{f(\boldsymbol{\theta})} \, g(\boldsymbol{\theta}) \tag{5.43}$$

$$= \int d\boldsymbol{\theta}\, d\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}}\, \overline{f(\boldsymbol{\theta})}\, g(\boldsymbol{\theta}) \tag{5.43a}$$

For $n = 1$, we have

$$(f, g) = \int d\theta\, d\overline{\theta}\, e^{\overline{\theta}\theta}\, \overline{f(\theta)}\, g(\theta) \tag{5.43b}$$

Writing f in terms of the basis $\{1, \theta\}$ gives

$$f(\theta) = a + b\,\theta \tag{5.43c}$$

which can also be deduced from the defining property [see eq(1.31) of §1.4]

$$\theta^2 = \overline{\theta}^2$$

Thus,

$$\overline{f(\theta)}\, f(\theta) = \left(\overline{a} + \overline{b}\,\overline{\theta}\right)(a + b\,\theta)$$
$$= \overline{a}\, a + \overline{b}\, b\, \overline{\theta}\, \theta + \overline{a}\, b\, \theta + \overline{b}\, a\, \overline{\theta}$$

and

$$e^{\overline{\theta}\theta} = 1 + \overline{\theta}\,\theta$$

$$\rightarrow \qquad e^{\overline{\theta}\theta}\, \overline{f(\theta)}\, f(\theta) = \left(1 + \overline{\theta}\,\theta\right)\left(\overline{a}\, a + \overline{b}\, b\, \overline{\theta}\, \theta + \overline{a}\, b\, \theta + \overline{b}\, a\, \overline{\theta}\right)$$
$$= \overline{a}\, a + \left(\overline{a}\, a + \overline{b}\, b\right)\overline{\theta}\,\theta + \overline{a}\, b\, \theta + \overline{b}\, a\, \overline{\theta}$$

Using [see eq(1.50) of §1.6]

$$\int d\theta = \int d\overline{\theta} = 0$$

$$\int d\theta\, A = \frac{\partial}{\partial\theta} A \qquad\qquad \int d\overline{\theta}\, A = \frac{\partial}{\partial\overline{\theta}} A$$

we have

$$(f, f) = \int d\theta\, d\overline{\theta}\left(\overline{a}\, a + \overline{b}\, b\right)\overline{\theta}\,\theta$$
$$= \int d\theta\, \left(\overline{a}\, a + \overline{b}\, b\right)\theta$$
$$= \left(\overline{a}\, a + \overline{b}\, b\right) \ge 0$$

i.e., the particular ordering of θ & $\overline{\theta}$ given in eq(5.43b) leads to the required positive norm.

Using

$$\int d\theta\, d\overline{\theta}\; e^{\overline{\theta}\theta} = \int d\theta\, d\overline{\theta}\left(1 + \overline{\theta}\,\theta\right) = \int d\theta\, d\overline{\theta}\; \overline{\theta}\,\theta = \int d\theta\, \theta = 1$$

we have

$$\left(\theta_k^{m_k},\, \theta_l^{m_l}\right) = \int \prod_{i=1}^{n} d\theta_i\, d\overline{\theta}_i\, \exp\left(\sum_{j=1}^{n} \overline{\theta}_j\, \theta_j\right) \overline{\theta}_k^{m_k}\, \theta_l^{m_l} \qquad\qquad (\, m_k\,,\ m_l = 0 \text{ or } 1\,)$$

If $k \neq l$, we have

$$\left(\theta_k^{m},\, \theta_l^{p}\right) = \int d\theta_k\, d\overline{\theta}_k\, d\theta_l\, d\overline{\theta}_l \left(1 + \overline{\theta}_k\, \theta_k + \overline{\theta}_l\, \theta_l\right) \overline{\theta}_k^{m}\, \theta_l^{p}$$
$$= \int d\theta_k\, d\overline{\theta}_k\, d\theta_l\, d\overline{\theta}_l \left(\overline{\theta}_k^{m}\, \theta_l^{p} + \delta_{m0}\, \overline{\theta}_k\, \theta_k\, \theta_l^{p} + \delta_{p0}\, \overline{\theta}_l\, \theta_l\, \overline{\theta}_k^{m}\right)$$
$$= 0$$

since every term has at least one $\int d\theta$ or $\int d\overline{\theta}$ factor.

Thus,

$$\left(\theta_k^m,\ \theta_l^p\right) \propto \delta_{k\,l}$$

If $k = l$, then

$$\left(\theta_k^m,\ \theta_k^p\right) = \int d\,\theta_k\, d\,\overline{\theta}_k \left(1 + \overline{\theta}_k\,\theta_k\right) \overline{\theta}_k^m\,\theta_k^p$$

$$= \int d\,\theta_k\, d\,\overline{\theta}_k \left(\overline{\theta}_k^m\,\theta_k^p + \delta_{m\,0}\,\delta_{p\,0}\,\overline{\theta}_k\,\theta_k\right)$$

$$= \delta_{m\,1}\,\delta_{p\,1} + \delta_{m\,0}\,\delta_{p\,0}$$

$$= \delta_{m\,p} \qquad \text{if} \qquad \theta_k^m\ \&\ \theta_k^p \neq 0\ \text{, i.e., } m,\ p \le 1$$

$$\to \qquad \left(\theta_k^m,\ \theta_l^p\right) = \begin{cases} \delta_{k\,l}\,\delta_{m\,p} & \text{if} \quad m,\ p \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$\left\{\ \theta_1^{m_1} \ldots \theta_n^{m_n}\ \middle|\ m_j = 0,\ 1\ \right\}$$

is an orthonormal basis for $\mathfrak{U}$ of dimension 2^n.

The δ-Function

Using eq(5.43c), we have

$$\int d\,\theta\,\theta\, f(\theta) = \int d\,\theta\,\theta\,(a + b\,\theta) = \int d\,\theta\,\theta\, a\ = a$$

$$= f(0)$$

$$\to \qquad \theta = \delta(\theta)$$

More generally,

$$\int d\,\theta\, d\,\overline{\theta}\ e^{\overline{\theta}\,\theta} f(\theta) = \int d\,\theta\, d\,\overline{\theta}\left(1 + \overline{\theta}\,\theta\right)(a + b\,\theta)$$

$$= \int d\,\theta\, d\,\overline{\theta}\left(a + b\,\theta + a\,\overline{\theta}\,\theta\right)\ = a$$

$$= f(0) \tag{5.44}$$

$$\to \qquad \int d\,\overline{\theta}\ e^{\overline{\theta}\,\theta} = \delta(\theta) \tag{5.44a}$$

Doing calculus by series expansion is tedious. A more pleasant approach is as follows.

In a series expansion of $f(\boldsymbol{\theta})$, each monomial term is linear in every θ_j. Therefore, $\dfrac{\partial\, f}{\partial\,\theta_j}$ must be free of θ_j. Thus, $\dfrac{\partial\, f}{\partial\,\theta_j}$ can be obtained in 2 steps. First, do the partial derivative using ordinary calculus formulas while observing the anti-symmetric rules. Secondly, set $\theta_j = 0$.

For example, using [see eq(1.50) of §1.6]

$$\int d\,\theta\, A = \left.\frac{\partial\, A}{\partial\,\theta}\right|_{\theta = 0}$$

we have

$$\int d\,\overline{\theta}\ e^{\overline{\theta}\,\theta} = \left.\theta\, e^{\overline{\theta}\,\theta}\right|_{\overline{\theta} = 0} = \left.e^{\overline{\theta}\,\theta}\,\theta\right|_{\overline{\theta} = 0} \qquad \left[\ \theta\left(\overline{\theta}\,\theta\right) = \left(\overline{\theta}\,\theta\right)\theta\ \right]$$

$$= \theta$$

$$\int d\theta\, d\bar{\theta}\; e^{\bar{\theta}\theta} f(\theta) = \int d\theta\; \theta f(\theta) = \left(f(\theta) - \theta \frac{\partial f}{\partial \theta} \right)_{\theta=0} = f(0)$$

Still more generally,

$$\begin{aligned}\int d\theta\, d\bar{\theta}\; e^{\bar{\theta}(\theta-\theta')} f(\theta) &= \int d\theta\,(\theta-\theta') f(\theta)\\ &= \left(f(\theta) - (\theta-\theta') \frac{\partial f(\theta)}{\partial \theta} \right)_{\theta=0}\\ &= f(0) + \theta' \left.\frac{\partial f(\theta)}{\partial \theta}\right|_{\theta=0}\\ &= f(\theta')\end{aligned}$$

$$\rightarrow \qquad \int d\bar{\theta}\; e^{\bar{\theta}(\theta-\theta')} = \delta(\theta-\theta')$$

Using $(\theta\,\theta')\,\theta'' = \theta''\,(\theta\,\theta')$, we have

$$\begin{aligned} d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}} &\equiv d\theta_1\, d\bar{\theta}_1 \ldots d\theta_n\, d\bar{\theta}_n\\ &= d\theta_n\, d\theta_1\, d\bar{\theta}_1 \ldots d\bar{\theta}_{n-1}\, d\bar{\theta}_n\\ &= d\theta_n\, d\theta_{n-1}\, d\theta_1\, d\bar{\theta}_1 \ldots d\bar{\theta}_{n-2}\, d\bar{\theta}_{n-1}\, d\bar{\theta}_n\\ &= d\theta_n \ldots d\theta_1\, d\bar{\theta}_1 \ldots d\bar{\theta}_n \end{aligned}$$

Hence,

$$\int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^{+}(\boldsymbol{\theta}-\boldsymbol{\theta}')} f(\boldsymbol{\theta})$$

$$= \int d\theta_n \ldots d\theta_1\, d\bar{\theta}_1 \ldots d\bar{\theta}_n\; e^{\boldsymbol{\theta}^{+}(\boldsymbol{\theta}-\boldsymbol{\theta}')} f(\boldsymbol{\theta})$$

$$= \int d\theta_n \ldots d\theta_1 \frac{\partial}{\partial \bar{\theta}_1} \ldots \frac{\partial}{\partial \bar{\theta}_n} \left[e^{\boldsymbol{\theta}^{+}(\boldsymbol{\theta}-\boldsymbol{\theta}')} f(\boldsymbol{\theta}) \right] \qquad \text{[See eqs(1.59-1.59a)]}$$

$$= \int d\theta_n \ldots d\theta_1 \frac{\partial}{\partial \bar{\theta}_1} \ldots \frac{\partial}{\partial \bar{\theta}_{n-1}} \left[(\theta_n - \theta_n')\, \left. e^{\boldsymbol{\theta}^{+}(\boldsymbol{\theta}-\boldsymbol{\theta}')} \right|_{\bar{\theta}_n=0} f(\boldsymbol{\theta}) \right]$$

$$= \int d\theta_n \ldots d\theta_1\, (\theta_1 - \theta_1') \ldots (\theta_n - \theta_n')\, f(\boldsymbol{\theta})$$

$$= \frac{\partial}{\partial \theta_n} \ldots \frac{\partial}{\partial \theta_1} \left. \left[(\theta_1 - \theta_1') \ldots (\theta_n - \theta_n')\, f(\boldsymbol{\theta}) \right] \right|_{\theta_1=\ldots=\theta_n=0}$$

$$= \left\{ \frac{\partial}{\partial \theta_n} \ldots \frac{\partial}{\partial \theta_2} \left[(\theta_2 - \theta_2') \ldots (\theta_n - \theta_n')\, f(\boldsymbol{\theta}) \right|_{\theta_1=0} + (-)^n (-\theta_1')(\theta_2 - \theta_2') \ldots (\theta_n - \theta_n') \left. \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_1} \right|_{\theta_1=0} \right] \right\} \Bigg|_{\theta_2=\ldots=\theta_n=0}$$

$$= \left\{ \frac{\partial}{\partial \theta_n} \ldots \frac{\partial}{\partial \theta_2} \left[(\theta_2 - \theta_2') \ldots (\theta_n - \theta_n')\, f(\boldsymbol{\theta}) \right|_{\theta_1=0} + (\theta_2 - \theta_2') \ldots (\theta_n - \theta_n')\, \theta_1' \left. \frac{\partial f(\boldsymbol{\theta})}{\partial \theta_1} \right|_{\theta_1=0} \right] \right\} \Bigg|_{\theta_2=\ldots=\theta_n=0}$$

$$\vdots$$

$$= f(0) + \theta_1' \ldots \theta_n' \left. \frac{\partial^n f(\boldsymbol{\theta})}{\partial \theta_n \ldots \partial \theta_1} \right|_{\boldsymbol{\theta}=0}$$

For any analytic function

$$f(\boldsymbol{\theta}) = \sum_{\boldsymbol{m}} c_{\boldsymbol{m}}\, \theta_1^{m_1} \ldots \theta_n^{m_n}$$

so that

$$\frac{\partial^k}{\partial \theta_k \ldots \partial \theta_1} f(\boldsymbol{\theta})\bigg|_{\boldsymbol{\theta}=0} = \begin{cases} c_{1\,\ldots 1} & \text{if} \quad k = n \\ 0 & \text{otherwise} \end{cases}$$

and the Taylor expansion of f about $\boldsymbol{\theta} = 0$ reduces to

$$\begin{aligned} f(\boldsymbol{\theta}) &= f(0) + c_{1\,\ldots 1}\, \theta_1 \ldots \theta_n \\ &= f(0) + \frac{\partial^n}{\partial \theta_n \ldots \partial \theta_1} f(\boldsymbol{\theta})\bigg|_{\boldsymbol{\theta}=0} \theta_1 \ldots \theta_n \end{aligned}$$

Hence,

$$\int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^+(\boldsymbol{\theta}-\boldsymbol{\theta}')} f(\boldsymbol{\theta}) = f(\boldsymbol{\theta}') \qquad (5.44b)$$

$$\rightarrow \qquad \int d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^+(\boldsymbol{\theta}-\boldsymbol{\theta}')} = \delta(\boldsymbol{\theta} - \boldsymbol{\theta}') \qquad (5.44c)$$

5.3.2. Operator Algebra

Warning: As in §5.2, differences between the matrix elements used by Zinn-Justin & ours persist. However, we'll no longer point them out individually.

The Identity as a Kernel

Using

$$f(\boldsymbol{\theta}) = \langle\, \boldsymbol{\theta} \mid f\,\rangle = \langle\, \theta_1 \ldots \theta_n \mid f\,\rangle$$

we can write eq(5.43a) as

$$\langle f \mid g\,\rangle = \int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^+\boldsymbol{\theta}} \langle f \mid \boldsymbol{\theta}\rangle\langle \boldsymbol{\theta} \mid g\,\rangle$$

The completeness of the basis $\mid \boldsymbol{\theta}\rangle$ is therefore given by

$$\int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^+\boldsymbol{\theta}} \mid \boldsymbol{\theta}\rangle\langle \boldsymbol{\theta} \mid = \mathcal{I} \qquad (5.45a)$$

Let

$$\langle\, \boldsymbol{\theta} \mid m_1 \ldots m_n\,\rangle = \frac{\theta_1^{m_1}}{\sqrt{m_1!}} \cdots \frac{\theta_n^{m_n}}{\sqrt{m_n!}} \qquad\qquad \langle\, m_1 \ldots m_n \mid \boldsymbol{\theta}\,\rangle = \frac{\overline{\theta}_n^{m_n}}{\sqrt{m_n!}} \cdots \frac{\overline{\theta}_1^{m_1}}{\sqrt{m_1!}}$$

then the completeness of the basis $\mid m_1 \ldots m_n\,\rangle$:

$$\sum_{m_1,\ldots,m_2} \mid m_1 \ldots m_n\,\rangle\langle\, m_1 \ldots m_n \mid = \mathcal{I} \qquad (\, m_j = 0,\, 1\,) \qquad (5.45b)$$

implies

$$\sum_{m_1,\ldots,m_2} \langle\, \boldsymbol{\theta} \mid m_1 \ldots m_n\,\rangle\langle\, m_1 \ldots m_n \mid \boldsymbol{\theta}'\,\rangle = \langle\, \boldsymbol{\theta} \mid \boldsymbol{\theta}'\,\rangle$$

Hence, using

$$\theta\,\theta'\,\theta'' = -\theta'\,\theta\,\theta'' = \theta'\,\theta''\,\theta$$

we have

$$\langle \boldsymbol{\theta} \mid \boldsymbol{\theta}' \rangle = \sum_{m_1,\ldots,m_2} \frac{\theta_1^{m_1}}{\sqrt{m_1!}} \cdots \frac{\theta_n^{m_n}}{\sqrt{m_n!}} \frac{\overline{\theta}_n'^{\,m_n}}{\sqrt{m_n!}} \cdots \frac{\overline{\theta}_1'^{\,m_1}}{\sqrt{m_1!}}$$

$$= \sum_{m_1,\ldots,m_2} \frac{\theta_1^{m_1}}{\sqrt{m_1!}} \cdots \frac{\theta_{n-2}^{m_{n-2}}}{\sqrt{m_{n-2}!}} \frac{(\theta_{n-1}\,\overline{\theta}_{n-1}')^{m_{n-1}}}{m_{n-1}!} \frac{(\theta_n\,\overline{\theta}_n')^{m_n}}{m_n!} \frac{\overline{\theta}_n'^{\,m_{n-2}}}{\sqrt{m_{n-2}!}} \cdots \frac{\overline{\theta}_1'^{\,m_1}}{\sqrt{m_1!}}$$

$$\vdots$$

$$= \sum_{m_1,\ldots,m_2} \frac{(\theta_1\,\overline{\theta}_1')^{m_1}}{m_1!} \cdots \frac{(\theta_n\,\overline{\theta}_n')^{m_n}}{m_n!}$$

$$= \exp\left(\sum_{i=1}^{n} \theta_i\,\overline{\theta}_i'\right) = \exp\left(-\sum_{i=1}^{n} \overline{\theta}_i'\,\theta_i\right) \qquad (5.45)$$

$$= e^{\boldsymbol{\theta}^T \overline{\boldsymbol{\theta}}'} = e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} \qquad (5.45c)$$

Note: in Zinn-Justin's notation, $\langle \boldsymbol{\theta} \mid \boldsymbol{\theta} \rangle = \mathcal{I}(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}})$.

Checking for consistency, we get from eq(5.45a)

$$\langle \boldsymbol{\theta}' \mid \boldsymbol{\theta}'' \rangle = \int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^{+}\boldsymbol{\theta}} \langle \boldsymbol{\theta}' \mid \boldsymbol{\theta} \rangle \langle \boldsymbol{\theta} \mid \boldsymbol{\theta}'' \rangle$$

Using eq(5.45), we have

$$e^{-\boldsymbol{\theta}''^{+}\boldsymbol{\theta}'} = \int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^{+}\boldsymbol{\theta}}\, e^{-\boldsymbol{\theta}^{+}\boldsymbol{\theta}'}\, e^{-\boldsymbol{\theta}''^{+}\boldsymbol{\theta}}$$

$$= \int d\,\boldsymbol{\theta}\, \delta\,(\boldsymbol{\theta} - \boldsymbol{\theta}')\, e^{-\boldsymbol{\theta}''^{+}\boldsymbol{\theta}} \qquad \text{[Eq(5.44c) used.]}$$

$$= e^{-\boldsymbol{\theta}''^{+}\boldsymbol{\theta}'}$$

Yet another way to check is via eqs(5.44 & 5.45a) as follows

$$f(\boldsymbol{\theta}) = \langle \boldsymbol{\theta} \mid f \rangle$$

$$= \int d\,\boldsymbol{\theta}'\, d\,\overline{\boldsymbol{\theta}}'\; e^{\boldsymbol{\theta}'^{+}\boldsymbol{\theta}'} \langle \boldsymbol{\theta} \mid \boldsymbol{\theta}' \rangle \langle \boldsymbol{\theta}' \mid f \rangle$$

$$= \int d\,\boldsymbol{\theta}'\, d\,\overline{\boldsymbol{\theta}}'\; e^{\boldsymbol{\theta}'^{+}(\boldsymbol{\theta}'-\boldsymbol{\theta})} f(\boldsymbol{\theta}')$$

$$= \int d\,\boldsymbol{\theta}'\, \delta(\boldsymbol{\theta}' - \boldsymbol{\theta})\, f(\boldsymbol{\theta}') \qquad \text{[Eq(5.44c) used.]}$$

$$= f(\boldsymbol{\theta})$$

Operator Algebra

Using the completeness eq(5.45a), any operator $O\left(\boldsymbol{\theta}, \frac{\partial}{\partial\,\boldsymbol{\theta}}\right)$ can be written as

$$O = \int d\,\boldsymbol{\theta}\, d\,\overline{\boldsymbol{\theta}}\; e^{\boldsymbol{\theta}^{+}\boldsymbol{\theta}} \int d\,\boldsymbol{\theta}'\, d\,\overline{\boldsymbol{\theta}}'\; e^{\boldsymbol{\theta}'^{+}\boldsymbol{\theta}'} \;\mid \boldsymbol{\theta} \rangle \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle \langle \boldsymbol{\theta}' \mid$$

where

$$\langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle = O\left(\boldsymbol{\theta}, \frac{\partial}{\partial\,\boldsymbol{\theta}}\right) \langle \boldsymbol{\theta} \mid \boldsymbol{\theta}' \rangle$$

$$= O\left(\boldsymbol{\theta}, \frac{\partial}{\partial\,\boldsymbol{\theta}}\right) e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} \qquad (5.48a)$$

Using

$$\frac{\partial}{\partial\,\theta_i}\sum_j \overline{\theta}'_j\,\theta_j = -\sum_j \overline{\theta}'_j\,\frac{\partial}{\partial\,\theta_i}\,\theta_j = -\overline{\theta}'_i$$

we have

$$\frac{\partial}{\partial\,\theta_i}e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} = \left(-\frac{\partial}{\partial\,\theta_i}\sum_j \overline{\theta}'_j\,\theta_j\right)e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} = \overline{\theta}'_i\,e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}}$$

Hence, for a normal ordered operator

$$\mathcal{N}\left(\boldsymbol{\theta},\,\frac{\partial}{\partial\,\boldsymbol{\theta}}\right) = \sum_{\boldsymbol{m},\boldsymbol{p}}\mathcal{N}_{\boldsymbol{m}\,\boldsymbol{p}}\,\theta_1^{m_1}\,\ldots\,\theta_n^{m_n}\,\frac{\partial^{p_1}}{\partial\,\theta_1^{p_1}}\,\ldots\,\frac{\partial^{p_n}}{\partial\,\theta_n^{p_n}}$$

we have

$$\langle\,\boldsymbol{\theta}\mid\mathcal{N}\mid\boldsymbol{\theta}'\,\rangle = \sum_{\boldsymbol{m},\boldsymbol{p}}\mathcal{N}_{\boldsymbol{m}\,\boldsymbol{p}}\,\theta_1^{m_1}\,\ldots\,\theta_n^{m_n}\,\overline{\theta}_1'^{\,p_1}\,\ldots\,\overline{\theta}_n'^{\,p_n}\,e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}}$$

$$= \mathcal{N}\left(\boldsymbol{\theta},\,\overline{\boldsymbol{\theta}}'\right)e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} \tag{5.48}$$

Hence,

$$\mathcal{N} = \int d\,\boldsymbol{\theta}\,d\,\overline{\boldsymbol{\theta}}\;e^{\boldsymbol{\theta}^{+}\boldsymbol{\theta}}\int d\,\boldsymbol{\theta}'\,d\,\overline{\boldsymbol{\theta}}'\;e^{\boldsymbol{\theta}'^{+}\boldsymbol{\theta}'}\,e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}}\;\mid\boldsymbol{\theta}\,\rangle\,\mathcal{N}\left(\boldsymbol{\theta},\,\overline{\boldsymbol{\theta}}'\right)\langle\,\boldsymbol{\theta}'\mid \quad (5.48b)$$

where

$$\mathcal{N}\left(\boldsymbol{\theta},\,\overline{\boldsymbol{\theta}}'\right) = \sum_{\boldsymbol{m},\boldsymbol{p}}\mathcal{N}_{\boldsymbol{m}\,\boldsymbol{p}}\,\theta_1^{m_1}\,\ldots\,\theta_n^{m_n}\,\overline{\theta}_1'^{\,p_1}\,\ldots\,\overline{\theta}_n'^{\,p_n}$$

is called the kernel of $\mathcal{N}$.

From eq(5.48a), we have

$$\begin{aligned}(O\,f)\,(\boldsymbol{\theta}) &= \langle\,\boldsymbol{\theta}\mid O\;\mid f\,\rangle\\
&= \int d\,\boldsymbol{\theta}'\,d\,\overline{\boldsymbol{\theta}}'\,e^{\boldsymbol{\theta}'^{+}\boldsymbol{\theta}'}\int d\,\boldsymbol{\theta}''\,d\,\overline{\boldsymbol{\theta}}''\;e^{\boldsymbol{\theta}''^{+}\boldsymbol{\theta}''}\,\langle\,\boldsymbol{\theta}\mid\boldsymbol{\theta}'\,\rangle\langle\,\boldsymbol{\theta}'\mid O\mid\boldsymbol{\theta}''\,\rangle\langle\,\boldsymbol{\theta}''\mid f\,\rangle\\
&= \int d\,\boldsymbol{\theta}'\,d\,\overline{\boldsymbol{\theta}}'\;e^{\boldsymbol{\theta}'^{+}(\boldsymbol{\theta}'-\boldsymbol{\theta})}\int d\,\boldsymbol{\theta}''\,d\,\overline{\boldsymbol{\theta}}''\;e^{\boldsymbol{\theta}''^{+}\boldsymbol{\theta}''}\,\langle\,\boldsymbol{\theta}'\mid O\mid\boldsymbol{\theta}''\,\rangle f(\boldsymbol{\theta}'')\\
&= \int d\,\boldsymbol{\theta}'\;\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\;\int d\,\boldsymbol{\theta}''\,d\,\overline{\boldsymbol{\theta}}''\;e^{\boldsymbol{\theta}''^{+}\boldsymbol{\theta}''}\,\langle\,\boldsymbol{\theta}'\mid O\mid\boldsymbol{\theta}''\,\rangle f(\boldsymbol{\theta}'')\\
&= \int d\,\boldsymbol{\theta}''\,d\,\overline{\boldsymbol{\theta}}''\;e^{\boldsymbol{\theta}''^{+}\boldsymbol{\theta}''}\,\langle\,\boldsymbol{\theta}\mid O\mid\boldsymbol{\theta}''\,\rangle f(\boldsymbol{\theta}'')\\
&= \int d\,\boldsymbol{\theta}'\,d\,\overline{\boldsymbol{\theta}}'\;e^{\boldsymbol{\theta}'^{+}\boldsymbol{\theta}'}\,\langle\,\boldsymbol{\theta}\mid O\mid\boldsymbol{\theta}'\,\rangle f(\boldsymbol{\theta}')\end{aligned} \tag{5.49a}$$

Using eq(5.48b), we have

$$(\mathcal{N}\,f)\,(\boldsymbol{\theta}) = \int d\,\boldsymbol{\theta}'\,d\,\overline{\boldsymbol{\theta}}'\;e^{\boldsymbol{\theta}'^{+}(\boldsymbol{\theta}'-\boldsymbol{\theta})}\,\mathcal{N}\left(\boldsymbol{\theta},\,\overline{\boldsymbol{\theta}}'\right)f(\boldsymbol{\theta}') \tag{5.49b}$$

which differs from Zinn-Justin's eq(5.49) by the extra factor $\langle\,\boldsymbol{\theta}\mid\boldsymbol{\theta}'\,\rangle = e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}}$.

Using the completeness eq(5.45a), we have

$$\langle\,\boldsymbol{\theta}\mid O_1\,O_2\mid\boldsymbol{\theta}'\,\rangle = \int d\,\boldsymbol{\theta}''\,d\,\overline{\boldsymbol{\theta}}''\;e^{\boldsymbol{\theta}''^{+}\boldsymbol{\theta}''}\,\langle\,\boldsymbol{\theta}\mid O_1\mid\boldsymbol{\theta}''\,\rangle\langle\,\boldsymbol{\theta}''\;\mid O_2\mid\boldsymbol{\theta}'\,\rangle \tag{5.50}$$

Trace

Similarly,

$$\mathrm{tr}\, O = \sum_{\boldsymbol{m}} \langle \boldsymbol{m} \mid O \mid \boldsymbol{m} \rangle \qquad (\boldsymbol{m} = \{m_1, \ldots, m_n\})$$
$$= \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\, e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}} \int d\boldsymbol{\theta}'\, d\bar{\boldsymbol{\theta}}'\, e^{\boldsymbol{\theta}'^+ \boldsymbol{\theta}'} \sum_{\boldsymbol{m}} \langle \boldsymbol{m} \mid \boldsymbol{\theta} \rangle \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle \langle \boldsymbol{\theta}' \mid \boldsymbol{m} \rangle$$

Assuming $\langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle$ contains only even powers of Grassmann variables so that it commutes with everything. Then

$$\mathrm{tr}\, O = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\, e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}} \int d\boldsymbol{\theta}'\, d\bar{\boldsymbol{\theta}}'\, e^{\boldsymbol{\theta}'^+ \boldsymbol{\theta}'} \sum_{\boldsymbol{m}} \langle \boldsymbol{m} \mid \boldsymbol{\theta} \rangle \langle \boldsymbol{\theta}' \mid \boldsymbol{m} \rangle \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle$$

Analogous to the derivation of eq(5.45), we have

$$\sum_{\boldsymbol{m}} \langle \boldsymbol{m} \mid \boldsymbol{\theta} \rangle \langle \boldsymbol{\theta}' \mid \boldsymbol{m} \rangle = \sum_{m_1, \ldots, m_2} \frac{\bar{\theta}_n^{m_n}}{\sqrt{m_n!}} \cdots \frac{\bar{\theta}_1^{m_1}}{\sqrt{m_1!}} \frac{\theta_1'^{m_1}}{\sqrt{m_1!}} \cdots \frac{\theta_n'^{m_n}}{\sqrt{m_n!}}$$
$$= \sum_{m_1, \ldots, m_2} \frac{(\bar{\theta}_n \theta_n')^{m_n}}{m_n!} \cdots \frac{(\bar{\theta}_1 \theta_1')^{m_1}}{m_1!}$$
$$= e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}'}$$

whereas [see eq(5.45)]

$$\sum_{\boldsymbol{m}} \langle \boldsymbol{\theta}' \mid \boldsymbol{m} \rangle \langle \boldsymbol{m} \mid \boldsymbol{\theta} \rangle = \langle \boldsymbol{\theta}' \mid \boldsymbol{\theta} \rangle = e^{-\boldsymbol{\theta}^+ \boldsymbol{\theta}'}$$

Hence,

$$\mathrm{tr}\, O = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\, e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}} \int d\boldsymbol{\theta}'\, d\bar{\boldsymbol{\theta}}'\, e^{\boldsymbol{\theta}'^+ \boldsymbol{\theta}'} e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}'} \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle$$
$$= \int d\boldsymbol{\theta}\, \delta(\boldsymbol{\theta} + \boldsymbol{\theta}') \int d\boldsymbol{\theta}'\, d\bar{\boldsymbol{\theta}}'\, e^{\boldsymbol{\theta}'^+ \boldsymbol{\theta}'} \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle \qquad \text{[Eq(5.44c) used.]}$$

The δ-function can be evaluated in two different manners:

$$\mathrm{tr}\, O = \int d\boldsymbol{\theta}'\, d\bar{\boldsymbol{\theta}}'\, e^{\boldsymbol{\theta}'^+ \boldsymbol{\theta}'} \langle -\boldsymbol{\theta}' \mid O \mid \boldsymbol{\theta}' \rangle$$
$$= \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\, e^{\boldsymbol{\theta}^+ \boldsymbol{\theta}} \langle -\boldsymbol{\theta} \mid O \mid \boldsymbol{\theta} \rangle \tag{5.51a}$$
$$= \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}'\, e^{-\boldsymbol{\theta}'^+ \boldsymbol{\theta}} \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle \tag{5.51}$$

Note that, in general, one cannot write

$$\mathrm{tr}\, O = \int d\boldsymbol{\theta}\, d\bar{\boldsymbol{\theta}}\, e^{\boldsymbol{\theta}^+ (\boldsymbol{\theta} + \boldsymbol{\theta}')} \int d\boldsymbol{\theta}'\, \delta(\boldsymbol{\theta}') \langle \boldsymbol{\theta} \mid O \mid \boldsymbol{\theta}' \rangle$$

because, by eq(5.48a), $\left\langle \boldsymbol{\theta} \,\middle|\, O\!\left(z, \frac{\partial}{\partial z}\right) \middle|\, \boldsymbol{\theta}' \right\rangle$ is a function of $\bar{\boldsymbol{\theta}}'$.

Hermitian Conjugation of Operators

From eq(5.48), we have

$$\overline{\langle \boldsymbol{\theta} \mid \mathcal{N} \mid \boldsymbol{\theta}' \rangle} = \langle \boldsymbol{\theta}' \mid \mathcal{N}^+ \mid \boldsymbol{\theta} \rangle = \overline{\mathcal{N}(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}')}\, e^{-\boldsymbol{\theta}^+ \boldsymbol{\theta}'}$$
$$\rightarrow \quad \langle \boldsymbol{\theta} \mid \mathcal{N}^+ \mid \boldsymbol{\theta}' \rangle = \overline{\mathcal{N}(\boldsymbol{\theta}', \bar{\boldsymbol{\theta}})}\, e^{-\boldsymbol{\theta}'^+ \boldsymbol{\theta}}$$
$$= \mathcal{N}^+(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}')\, e^{-\boldsymbol{\theta}'^+ \boldsymbol{\theta}} \qquad \text{[By eq(5.48)]}$$

$\therefore \qquad \mathcal{N}^{+}(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}') = \overline{\mathcal{N}(\boldsymbol{\theta}', \overline{\boldsymbol{\theta}})}$ as expected.

5.3.3. Fermion States & Path Integrals

One-state Hamiltonian & Fermions

A general 1-fermion state is given by

$$\psi(\theta) = \psi_0 + \psi_1 \theta \qquad \psi(\theta) \in \mathfrak{U}$$

where

$\theta^m \propto$ state occupied by m fermions.

Let the energy of the state be $\hbar\omega$. Then

$$H_0 \theta^m = E_m \theta^m \qquad \text{with} \qquad E_m = \begin{cases} 0 & \text{if } m = 0 \\ \hbar\omega & \text{if } m = 1 \end{cases}$$

Borrowing the mathematics of developed for the harmonic oscillator, we write

$$H_0 = \hbar\omega\theta \frac{\partial}{\partial\theta} \tag{5.52}$$

We stress that physically, eq(5.52) represents any system with only two possible energies, 0 or $\hbar\omega$. It has nothing to do with oscillators.

The kernel of H_0 is [c.f. eq(5.48)]

$$\begin{aligned}
\langle \theta \mid H_0 \mid \theta' \rangle &= \hbar\omega\theta \frac{\partial}{\partial\theta} \langle \theta \mid \theta' \rangle \\
&= \hbar\omega\theta \frac{\partial}{\partial\theta} e^{-\overline{\theta}'\theta} \\
&= \hbar\omega\theta \left(-\frac{\partial}{\partial\theta} (\overline{\theta}'\theta) \right) e^{-\overline{\theta}'\theta} = \hbar\omega\theta\,\overline{\theta}' \left(\frac{\partial}{\partial\theta} \theta \right) e^{-\overline{\theta}'\theta} \\
&= \hbar\omega\theta\overline{\theta}' e^{-\overline{\theta}'\theta} = \hbar\omega\theta\overline{\theta}' \left(1 - \overline{\theta}'\theta \right) = \hbar\omega\theta\overline{\theta}' \\
&= -\hbar\omega\overline{\theta}'\theta
\end{aligned} \tag{5.52a}$$

For the evolution operator, we have

$$U_0(t) = e^{-t H_0/\hbar} = \sum_{k=0}^{\infty} \frac{(-\omega t)^k}{k!} \left(\theta \frac{\partial}{\partial\theta} \right)^k$$

$$\rightarrow \qquad \langle \theta \mid U_0(t) \mid \theta' \rangle = \sum_{k=0}^{\infty} \frac{(-\omega t)^k}{k!} \left(\theta \frac{\partial}{\partial\theta} \right)^k e^{-\overline{\theta}'\theta}$$

$$= \sum_{k=0}^{\infty} \frac{(-\omega t)^k}{k!} P^k e^{-\rho} \tag{5.52b}$$

where

$$P = \theta \frac{\partial}{\partial\theta} \qquad \rho = \overline{\theta}'\theta$$

If we were dealing with n oscillators, then

$$\boldsymbol{\theta} = \{\theta_1 \ldots \theta_n\} \qquad \& \qquad \rho = \boldsymbol{\theta}'^{+} \boldsymbol{\theta}$$

Owing to the cross terms, we have

$$\rho^m \neq 0 \qquad \forall\ m \leq n$$
$$\rho^m = 0 \qquad \forall\ m > n$$

For the example of $n = 2$:

$$\boldsymbol{\theta} = \{\theta_1,\ \theta_2\} \qquad \rightarrow \qquad \rho = \overline{\theta}_1'\, \theta_1 + \overline{\theta}_2'\, \theta_2$$
$$\rho^2 = 2\,\overline{\theta}_1'\, \theta_1\, \overline{\theta}_2'\, \theta_2 \qquad\qquad \rho^3 = 0$$

With this in mind, we have

$$P\,\rho = \theta \frac{\partial}{\partial\,\theta}\left(\overline{\theta}'\,\theta\right) = -\theta\,\overline{\theta}' \frac{\partial}{\partial\,\theta}\,\theta = -\theta\,\overline{\theta}' \ = \rho$$

$$\rightarrow \qquad P^k\,\rho = \rho \qquad \forall\ k$$

$$P\,\rho^m = \theta \frac{\partial}{\partial\,\theta}\left(\rho\,\rho^{m-1}\right) = \theta\left[\left(\frac{\partial}{\partial\,\theta}\,\rho\right)\rho^{m-1} + \rho\,\frac{\partial}{\partial\,\theta}\,\rho^{m-1}\right]$$
$$= (P\,\rho)\,\rho^{m-1} + \theta\,\rho\,\frac{\partial}{\partial\,\theta}\,\rho^{m-1}$$
$$= \rho^m + \rho\,P\rho^{m-1}$$
$$= \rho^m + \rho\left(\rho^{m-1} + \rho\,P\,\rho^{m-2}\right)$$
$$\vdots$$
$$= m\,\rho^m$$

$$\rightarrow \qquad P^n\,\rho^m = m^n\,\rho^m$$

Hence, eq(5.52b) becomes

$$\langle\,\boldsymbol{\theta}\mid U_0(t)\mid \boldsymbol{\theta}'\,\rangle = \sum_{k=0}^{\infty}\sum_{m=0}^{n} \frac{(-\omega\,t\,)^k}{k!\,m!}\,(-)^m\,P^k\,\rho^m$$
$$= \sum_{k=0}^{\infty}\sum_{m=0}^{n} \frac{(-\omega\,t\,)^k}{k!\,m!}\,(-)^m\,m^k\,\rho^m$$
$$= \sum_{m=0}^{n} \frac{e^{-m\,\omega\,t}}{m!}\,(-\rho)^m$$
$$= \sum_{m=0}^{\infty} \frac{\left(-\rho\,e^{-\,\omega\,t}\,\right)^m}{m!} \qquad\qquad [\ \rho^m = 0\ \forall\ m > n \text{ used}\ ]$$
$$= e^{-\boldsymbol{\theta}'^{+}\,\boldsymbol{\theta}\,e^{-\omega\,t}} \qquad\qquad (5.53a)$$

Back to our case of a single oscillator,

$$\langle\,\theta\mid U_0(t)\mid \theta'\,\rangle = e^{-\overline{\theta}'\,\theta\,e^{-\omega\,t}} \qquad\qquad (5.53)$$

For the partition function,

$$\mathcal{Z}_0(\beta) = \operatorname{tr} U_0(\hbar\,\beta)$$
$$= \int d\,\theta\,d\,\overline{\theta}'\ e^{-\overline{\theta}'\,\theta}\,\langle\,\theta\mid U_0(\hbar\beta)\mid \theta'\,\rangle \qquad\qquad [\text{ Eq(5.51) used }]$$
$$= \int d\,\theta\,d\,\overline{\theta}'\ \exp\left[-\,\overline{\theta}'\,\theta\left(1 + e^{-\beta\,\hbar\,\omega}\right)\right]$$
$$= \int d\,\theta\,\delta\left[\theta\left(1 + e^{-\beta\,\hbar\,\omega}\right)\right]$$

Alternatively, using eq(5.51a), we have

$$\mathcal{Z}_0(\beta) = \int d\,\theta\,d\,\overline{\theta}\,e^{\overline{\theta}\,\theta}\,\langle\,-\theta\mid U_0(\hbar\beta)\mid \theta\,\rangle$$

$$= \int d\theta\, d\bar{\theta}\, \exp\left[\bar{\theta}\,\theta\left(1+e^{-\beta\hbar\omega}\right)\right]$$
$$= \int d\theta\, \delta\left[\theta\left(1+e^{-\beta\hbar\omega}\right)\right]$$

Using eq(1.58), we have

$$\mathcal{Z}_0(\beta) = \int d\theta'\, \delta(\theta') \left| \frac{\partial\theta'}{\partial\theta} \right|$$

where

$$\theta' = \theta\left(1+e^{-\beta\hbar\omega}\right)$$

Hence,

$$\mathcal{Z}_0(\beta) = \left(1+e^{-\beta\hbar\omega}\right)\int d\theta'\,\delta(\theta')$$
$$= 1+e^{-\beta\hbar\omega} \tag{5.54}$$

which agrees with the well-known result obtained from statistical mechanics using commuting variables.

Remark

$$\left\{\theta, \frac{\partial}{\partial\theta}\right\} f \equiv \theta\frac{\partial}{\partial\theta} f + \frac{\partial}{\partial\theta}(\theta f) = \theta\frac{\partial}{\partial\theta} f + f - \theta\frac{\partial}{\partial\theta} f = f \qquad \forall\, f \in \mathfrak{U}$$

$$\to \qquad \left\{\theta, \frac{\partial}{\partial\theta}\right\} = 1$$

Connection with the 2nd quantization formulism is then obvious. Setting

$$\theta = a^+ \qquad\qquad \frac{\partial}{\partial\theta} = a$$

we have

$$a^2 = a^{+2} = 0$$
$$\left\{a, a^+\right\} = a\,a^+ + a^+\,a = 1 \tag{5.55}$$

Eq(5.52) becomes

$$H_0 = \hbar\,\omega\, a^+\, a \tag{5.56}$$

Path Integral

This section is the fermionic adaptation of the derivation in §5.2.1.

For small time ε, eq(5.53) becomes

$$\langle\,\theta\mid U_0(\varepsilon)\mid\theta'\,\rangle = \exp\left[-\bar{\theta}'\,\theta\,(1-\omega\,\varepsilon) + O\left(\varepsilon^2\right)\right] \tag{5.57}$$

$$\to \qquad \langle\,\theta_{k+1}\mid U_0(\varepsilon)\mid\theta_k\,\rangle = \exp\left[-\bar{\theta}_k\,\theta_{k+1}\,(1-\omega\,\varepsilon) + O\left(\varepsilon^2\right)\right]$$

Using eq(5.50), the group property becomes

$$\langle\,\theta''\mid U_0(t'', t')\mid\theta'\,\rangle = \int d\theta''\, d\bar{\theta}''\, e^{\bar{\theta}''\theta''}\langle\,\theta\mid U_0(t, t'')\mid\theta''\,\rangle\langle\,\theta''\mid U_0(t'', t')\mid\theta'\,\rangle$$

Setting

$$\varepsilon = \frac{t''-t'}{n} \qquad\qquad t_k = t' + k\,\varepsilon \qquad\qquad n = \text{odd}$$

$$\theta_k = \theta(t_k) \qquad \bar{\theta}_k = \bar{\theta}(t_k)$$
$$\theta_0 = \theta' \qquad \theta_n = \theta'' \tag{5.60}$$

we have

$$\langle \theta'' \mid U_0(t'', t') \mid \theta' \rangle = \lim_{n\to\infty} \int \prod_{k=1}^{n-1} d\,\theta_k\, d\,\bar{\theta}_k\; e^{\bar{\theta}_k \theta_k} \langle \theta_{k+1} \mid U_0(t_{k+1}, t_k) \mid \theta_k \rangle$$
$$\times \langle \theta_1 \mid U_0(t_1, t_0) \mid \theta_0 \rangle$$
$$= \lim_{n\to\infty} \int \prod_{k=1}^{n-1} d\,\theta_k\, d\,\bar{\theta}_k\, e^{-S_\varepsilon\left(\theta,\bar{\theta}\right)} \tag{5.58}$$

where

$$S_\varepsilon\left(\theta, \bar{\theta}\right) = -\sum_{k=1}^{n-1} \bar{\theta}_k\, \theta_k + \sum_{k=0}^{n-1} \bar{\theta}_k\, \theta_{k+1}\,(1 - \omega\,\varepsilon)$$
$$= \sum_{k=1}^{n-1} \bar{\theta}_k(\theta_{k+1} - \theta_k) + \bar{\theta}_0\,\theta_1 - \varepsilon\,\omega \sum_{k=0}^{n-1} \bar{\theta}_k\,\theta_{k+1} \tag{5.59}$$
$$= \bar{\theta}_0\,\theta_1 + \varepsilon\left(\sum_{k=1}^{n-1} \bar{\theta}_k \left(\frac{\theta_{k+1} - \theta_k}{\varepsilon}\right) - \omega \sum_{k=0}^{n-1} \bar{\theta}_k\,\theta_{k+1}\right)$$

As $\varepsilon \to 0$ & $n \to \infty$, eq(5.58) becomes a path integral

$$\langle \theta'' \mid U_0(t'', t') \mid \theta' \rangle = \int_{\theta(t')=\theta'}^{\theta(t'')=\theta''} \left[d\,\theta\,(t)\, d\,\bar{\theta}\,(t)\right] e^{-S_0\left(\theta,\bar{\theta}\right)} \tag{5.61}$$

with

$$\left[d\,\theta\,(t)\, d\,\bar{\theta}\,(t)\right] = \prod_{k=1}^{n-1} d\,\theta_k\, d\,\bar{\theta}_k$$

$$S_0\left(\theta, \bar{\theta}\right) = \bar{\theta}(t')\,\theta(t') + \int_{t'}^{t''} d\,t\;\; \bar{\theta}(t)\left[\dot{\theta}(t) - \omega\,\theta(t)\right] \tag{5.62}$$

Partition Function

Eqs(5.51a & 5.58-9) give

$$\mathcal{Z}_0(\beta) = \mathrm{tr}\; U_0(\hbar\beta/2\,,\, -\hbar\beta/2)$$
$$= \int d\,\theta_0\, d\,\bar{\theta}_0\;\, e^{\bar{\theta}_0\,\theta_0} \langle -\theta_0 \mid U_0(\hbar\beta/2\,,\, -\hbar\beta/2) \mid \theta_0 \rangle$$
$$= \int \prod_{k=0}^{n-1} d\,\theta_k\, d\,\bar{\theta}_k\;\, e^{-S_\varepsilon\left(\theta,\bar{\theta}\right)}$$

with

$$\theta(\hbar\beta/2) = \theta_n = -\theta_0 \qquad \& \qquad \theta(-\hbar\beta/2) = \theta_0$$

$$S_\varepsilon\left(\theta, \bar{\theta}\right) = -\bar{\theta}_0\,\theta_0 + \bar{\theta}_0\,\theta_1 + \sum_{k=1}^{n-1} \bar{\theta}_k(\theta_{k+1} - \theta_k) - \varepsilon\,\omega \sum_{k=0}^{n-1} \bar{\theta}_k\,\theta_{k+1}$$
$$= \sum_{k=0}^{n-1} \left[\bar{\theta}_k(\theta_{k+1} - \theta_k) - \varepsilon\,\omega\,\bar{\theta}_k\,\theta_{k+1}\right]$$

As $\varepsilon \to 0$ & $n \to \infty$, we have

$$\mathcal{Z}_0(\beta) = \int \left[d\,\theta(t)\, d\,\bar{\theta}(t)\right] e^{-S_0\left(\theta,\bar{\theta}\right)} \tag{5.64}$$

with

$$\left[d\,\theta(t)\,d\,\overline{\theta}(t)\right] = \prod_{k=0}^{n-1} d\,\theta_k\,d\,\overline{\theta}_k \qquad \& \qquad \theta(\hbar\beta/2) = -\theta(-\hbar\beta/2)$$

$$S_0\left(\theta,\,\overline{\theta}\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\;\;\overline{\theta}(t)\left[\dot{\theta}(t) - \omega\,\theta(t)\right] \tag{5.65}$$

Note that the boundary condition is anti-periodic.

General Gaussian Integral

This section is the fermionic adaptation of the derivation in §“Generating Functional” of §5.2.1.

Let

$$H = H_0 - \hbar\left(\overline{\eta}(t)\,\theta + \frac{\partial}{\partial\,\theta}\,\eta(t)\right)$$
$$= \hbar\left(\omega\,\theta\,\frac{\partial}{\partial\,\theta} - \overline{\eta}(t)\,\theta - \frac{\partial}{\partial\,\theta}\,\eta(t)\right)$$

where η & $\overline{\eta} \in \mathfrak{U}$ and

$$\mathcal{Z}_G(\beta) = \int\left[d\,\theta(t)\,d\,\overline{\theta}(t)\right] e^{-S_G\left(\theta,\overline{\theta}\right)}$$

$$S_G\left(\theta,\,\overline{\theta}\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\,\left\{\,\overline{\theta}(t)\left[\dot{\theta}(t) - \omega\,\theta(t)\right] - \overline{\eta}(t)\,\theta(t) - \overline{\theta}(t)\,\eta(t)\right\} \tag{5.66}$$

Note that each term in H is still quadratic in Grassmann variables so that the trace formulae eqs(5.51-a) remain applicable to U_G.

The classical equation of motion (with $\frac{\partial}{\partial\theta} \propto$ momentum) is given by

$$\dot{\overline{\theta}}_c = -\frac{1}{\hbar}\left[\frac{\partial}{\partial\,\theta},\,H\right]_{\theta=\theta_c,\;\partial/\partial\theta=\overline{\theta}_c}$$
$$= -\left[\frac{\partial}{\partial\,\theta},\,\omega\,\theta\,\frac{\partial}{\partial\,\theta} - \overline{\eta}(t)\,\theta - \frac{\partial}{\partial\,\theta}\,\eta(t)\right]_c$$

Using

$$\frac{\partial^2}{\partial\,\theta^2} f = \frac{\partial^2}{\partial\,\theta^2}(a + b\,\theta) = 0$$

we have

$$\left[\frac{\partial}{\partial\,\theta},\,\theta\,\frac{\partial}{\partial\,\theta}\right] f = \frac{\partial}{\partial\,\theta}\left(\theta\,\frac{\partial}{\partial\,\theta} f\right) - \theta\,\frac{\partial}{\partial\,\theta}\left(\frac{\partial}{\partial\,\theta} f\right) = \frac{\partial}{\partial\,\theta} f$$

$$\rightarrow \qquad \left[\frac{\partial}{\partial\,\theta},\,\theta\,\frac{\partial}{\partial\,\theta}\right] = \frac{\partial}{\partial\,\theta}$$

$$\left[\frac{\partial}{\partial\,\theta},\;\overline{\eta}(t)\,\theta\right] f = \frac{\partial}{\partial\,\theta}\left(\overline{\eta}(t)\,\theta f\right) - \overline{\eta}(t)\,\theta\,\frac{\partial}{\partial\,\theta} f \;= -\overline{\eta}(t)\,\frac{\partial}{\partial\,\theta}\left(\theta f\right) - \overline{\eta}(t)\,\theta\,\frac{\partial}{\partial\,\theta} f$$
$$= -\overline{\eta}(t)\left(f - \theta\,\frac{\partial}{\partial\,\theta} f + \theta\,\frac{\partial}{\partial\,\theta} f\right) = -\overline{\eta}(t)\,f$$

$$\to \quad \left[\frac{\partial}{\partial\,\theta},\ \overline{\eta}(t)\,\theta\right] = -\overline{\eta}(t)$$

Therefore, the classical equation of motion is

$$\dot{\overline{\theta}}_c = -\,\omega\,\overline{\theta}_c\ -\ \overline{\eta}(t) \qquad (5.67b)$$

Note: In our formulism, Zinn-Justin's eq(5.67a) should not be used in the following calculations.

The green's function associated with eq(5.67b) is

$$\dot{\Delta}(t) +\ \omega\,\Delta(t) = \delta(t) \qquad (5.67c)$$

so that

$$\overline{\theta}_c(t) = -\int_{t'}^{t''} d\,u\,\Delta(t-u)\,\overline{\eta}(u) \qquad (5.68)$$

Checking for consistency, we have

$$\dot{\overline{\theta}}_c(t) = -\int_{t'}^{t''} d\,u\,\dot{\Delta}(t-u)\,\overline{\eta}(u)$$

$$= -\int_{t'}^{t''} d\,u\,[\,-\omega\,\Delta(t-u) + \delta(t-u)\,]\,\overline{\eta}(u)$$

$$= -\omega\,\overline{\theta}_c(t) - \overline{\eta}(t) \qquad \text{as required.}$$

Δ is an ordinary complex function satisfying the anti-periodic B.C. [see eq(5.65)]

$$\Delta(\beta\,\hbar\,\omega/2) = -\Delta(-\beta\,\hbar\,\omega/2)$$

Following the derivation of eq(5.30), we have

$$\Delta(t) = \begin{cases} c\,e^{-\omega\,t} & \text{for } t > 0 \\ c'\,e^{-\omega\,t} & \text{for } t < 0 \end{cases}$$

where the discontinuity at $t = 0$ & anti-periodic B.C. give

$$c - c' = 1 \qquad \& \qquad c\,e^{-\beta\,\hbar\,\omega/2} = -c'\,e^{\beta\,\hbar\,\omega/2}$$

$$\to \quad c = \frac{1}{1\ +\ e^{-\beta\,\hbar\,\omega}} = \frac{e^{\beta\,\hbar\,\omega/2}}{2\cosh(\,\beta\,\hbar\,\omega\,/2\,)}$$

$$c' = -\frac{e^{-\beta\,\hbar\,\omega}}{1\ +\ e^{-\beta\,\hbar\,\omega}} = -\frac{e^{-\beta\,\hbar\,\omega/2}}{2\cosh(\,\beta\,\hbar\,\omega\,/2\,)}$$

$$\therefore \quad \Delta(t) = \frac{e^{-\omega\,t}}{1\ +\ e^{-\beta\,\hbar\,\omega}} \begin{cases} 1 & \text{for } t > 0 \\ -e^{-\beta\,\hbar\,\omega} & \text{for } t < 0 \end{cases} \qquad (5.69a)$$

$$= \frac{e^{-\omega\,t}}{2\left(1\ +\ e^{-\beta\,\hbar\,\omega}\right)}\left[\,\epsilon(t)\left(1 + e^{-\beta\,\hbar\,\omega}\right) + 1 - e^{-\beta\,\hbar\,\omega}\,\right]$$

where

$$\epsilon(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

$$\to \quad \Delta(t) = \frac{e^{-\omega\,t}}{2}[\,\epsilon(t) + \tanh(\beta\,\hbar\,\omega\,/2\,)\,] \qquad (5.69)$$

Setting

$$\theta(t) = \theta_c(t) + r(t)$$

with

$$\theta_c(\beta\hbar\omega/2) = \theta(\beta\hbar\omega/2) = -\theta(-\beta\hbar\omega/2) = -\theta_c(-\beta\hbar\omega/2)$$

$$\rightarrow \quad r(\beta\hbar\omega/2) = r(-\beta\hbar\omega/2) = 0$$

we have, from eq(5.66)

$$\begin{aligned} S_G(\theta,\overline{\theta}) &= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \left\{ \left[\dot{\overline{\theta}}(t) + \omega\,\overline{\theta}(t)\right]\theta(t) + \overline{\eta}(t)\,\theta(t) + \overline{\theta}(t)\,\eta(t)\right\} \\ &= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \left\{ \left(\dot{\overline{\theta}}_c + \dot{\overline{r}} + \omega\,\overline{\theta}_c + \omega\,\overline{r}\right)(\theta_c + r) + \overline{\eta}\,(\theta_c + r) + \left(\overline{\theta}_c + \overline{r}\right)\eta\right\} \\ &= S_0(r,\overline{r}) + S_G(\theta_c,\overline{\theta}_c) + S_L(\theta_c, r)\end{aligned}$$

where

$$S_0(r,\overline{r}) = -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\left(\dot{\overline{r}} + \omega\,\overline{r}\right) r$$

$$\begin{aligned} S_G(\theta_c,\overline{\theta}_c) &= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\left\{\left(\dot{\overline{\theta}}_c + \omega\,\overline{\theta}_c\right)\theta_c + \overline{\eta}\,\theta_c + \overline{\theta}_c\,\eta\right\} \\ &= -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\;\overline{\theta}_c\,\eta \\ &= \int_{-\hbar\beta/2}^{\hbar\beta/2} dt\; du\,\Delta(t-u)\,\overline{\eta}(u)\,\eta(t) \end{aligned} \qquad (5.70)$$

$$S_L(\theta_c, r) = -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\left\{\left(\dot{\overline{\theta}}_c + \omega\,\overline{\theta}_c\right) r + \left(\dot{\overline{r}} + \omega\,\overline{r}\right)\theta_c + \overline{\eta}\,r + \overline{r}\,\eta\right\} = 0$$

Hence,

$$\begin{aligned} \mathcal{Z}_G(\beta) &= \int [\,dr(t)\,d\overline{r}(t)\,]\,e^{-S_0(r,\overline{r})}\,e^{-S_G(\theta_c,\overline{\theta}_c)} \\ &= \mathcal{Z}_0(\beta)\,e^{-S_G(\theta_c,\overline{\theta}_c)} \\ &= \left(1 + e^{-\beta\hbar\omega}\right)\exp\left[-\int_{-\hbar\beta/2}^{\hbar\beta/2} dt\,du\,\Delta(t-u)\,\overline{\eta}(u)\,\eta(t)\right] \end{aligned} \qquad (5.71)$$

Since each term in S_G is quadratic in Grassmann variables, we get from eq(5.66)

$$\begin{aligned} \frac{\delta}{\delta\,\overline{\eta}(t_1)}\,e^{-S_G(\theta,\overline{\theta})} &= -\frac{\delta\,S_G(\theta,\overline{\theta})}{\delta\,\overline{\eta}(t_1)}\,e^{-S_G(\theta,\overline{\theta})} = \left[\frac{\delta}{\delta\,\overline{\eta}(t_1)}\int dt\,\overline{\eta}(t)\,\theta(t)\right]e^{-S_G(\theta,\overline{\theta})} \\ &= \int dt\left[\frac{\delta}{\delta\,\overline{\eta}(t_1)}\,\overline{\eta}(t)\right]\theta(t)\,e^{-S_G(\theta,\overline{\theta})} \\ &= \int dt\,\delta\,(t - t_1)\,\theta(t)\,e^{-S_G(\theta,\overline{\theta})} \\ &= \theta(t_1)\,e^{-S_G(\theta,\overline{\theta})} \end{aligned}$$

$$\begin{aligned} \frac{\delta}{\delta\,\eta(t_2)}\,\frac{\delta}{\delta\,\overline{\eta}(t_1)}\,e^{-S_G(\theta,\overline{\theta})} &= \theta(t_1)\left(-\frac{\delta\,S_G(\theta,\overline{\theta})}{\delta\,\eta(t_2)}\right)e^{-S_G(\theta,\overline{\theta})} \\ &= \theta(t_1)\left[\frac{\delta}{\delta\,\eta(t_2)}\int dt\,\overline{\theta}(t)\,\eta(t)\right]e^{-S_G(\theta,\overline{\theta})} \\ &= \theta(t_1)\left[-\int dt\,\overline{\theta}(t)\,\frac{\delta}{\delta\,\eta(t_2)}\,\eta(t)\right]e^{-S_G(\theta,\overline{\theta})} \end{aligned}$$

$$= -\theta(t_1) \int d\,t\, \overline{\theta}(t)\, \delta\,(t - t_2) \Big] e^{-S_G(\theta, \overline{\theta})}$$

$$= -\theta(t_1)\, \overline{\theta}\,(t_2)\, e^{-S_G(\theta,\overline{\theta})}$$

Let

$$X = \exp\Big[-\int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, d\,u\, \Delta(t-u)\, \overline{\eta}(u)\, \eta(t) \Big]$$

then, since Δ is just a function,

$$\langle \overline{\theta}(t_2)\, \theta(t_1) \rangle_0 = \frac{1}{\mathcal{Z}_0(\beta)} \int [\, d\,\theta(t)\, d\,\overline{\theta}(t)\,]\, \overline{\theta}(t_2)\, \theta(t_1)\, e^{-S_0(\theta,\overline{\theta})}$$

$$= -\frac{1}{\mathcal{Z}_0(\beta)} \frac{\delta}{\delta\,\eta(t_2)} \frac{\delta}{\delta\,\overline{\eta}(t_1)} \mathcal{Z}_G(\beta) \Big|_{\eta=\overline{\eta}=0}$$

$$= -\frac{\delta}{\delta\,\eta(t_2)} \frac{\delta}{\delta\,\overline{\eta}(t_1)} X \Big|_{\eta=\overline{\eta}=0}$$

$$= \frac{\delta}{\delta\,\eta(t_2)} \Big\{ \Big[\int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, \Delta(t-t_1)\, \eta(t) \Big] X \Big\} \Big|_{\eta=\overline{\eta}=0}$$

$$= \Big\{ \Delta(t_2 - t_1)\, X - \Big[\int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, \Delta(t-t_1)\, \eta(t) \Big] \frac{\delta}{\delta\,\eta(t_2)} X \Big\} \Big|_{\eta=\overline{\eta}=0}$$

$$= \Delta(t_2 - t_1) \qquad (5.72)$$

Remark

See Zinn-Justin's text.

The Gaussian Integration & the $\epsilon(0)$ Problem

This section is the fermionic adaptation of the derivation in §"Verification" of §5.2.1.

From eq(5.65), we have

$$\frac{\partial\, S_0}{\partial\,\omega} = -\int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, \overline{\theta}(t)\, \theta(t)$$

$$\rightarrow \quad \frac{\partial}{\partial\,\omega} \ln \mathcal{Z}_0(\beta) = -\frac{1}{\mathcal{Z}_0(\beta)} \int [\, d\,\theta(t)\, d\,\overline{\theta}(t)\,] \frac{\partial\, S_0}{\partial\,\omega} e^{-S_0(\theta,\overline{\theta})} \qquad \text{[Eq(5.65) used.]}$$

$$= \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, \langle \overline{\theta}(t)\, \theta(t) \rangle_0$$

$$= \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\, \Delta(0)$$

$$= \hbar\,\beta\,\Delta(0) \qquad (5.73a)$$

which differs from the bosonic case eq(5.32a) by a minus sign.

Following the derivation of eq(5.33) in §5.2.1, we have

$$\Delta(0) = \frac{1}{2} [\, \epsilon(0) + \tanh(\beta\,\hbar\,\omega\,/\,2\,)\,] \qquad \text{[Eq(5.69) used.]}$$

$$\ln \mathcal{Z}_0(\beta) = \frac{1}{2}\hbar\beta\omega\epsilon(0) + \ln\cosh(\beta\hbar\omega/2) + C$$

Since the ground (lowest energy) state of the system is $E_0 = 0$, we have, for $\beta \to \infty$,

$$\ln \mathcal{Z}_0(\beta) = -\beta E_0 = 0 = \frac{1}{2}\hbar\beta\omega\epsilon(0) + \frac{1}{2}\hbar\beta\omega - \ln 2 + C \qquad \left(\ln\cosh x \to \ln\frac{e^x}{2} = x - \ln 2\right)$$

$$\to \qquad C = -\frac{1}{2}\beta\hbar\omega - \frac{1}{2}\hbar\beta\omega\epsilon(0) + \ln 2$$

$$\therefore \qquad \ln \mathcal{Z}_0(\beta) = -\frac{1}{2}\hbar\beta\omega + \ln[2\cosh(\beta\hbar\omega/2)]$$

$$= \ln\left[e^{-\hbar\beta\omega/2}\left(e^{\hbar\beta\omega/2} + e^{-\hbar\beta\omega/2}\right)\right]$$

$$= \ln\left(1 + e^{-\hbar\beta\omega}\right)$$

$$\to \qquad \mathcal{Z}_0(\beta) = 1 + e^{-\hbar\beta\omega} \tag{5.73}$$

Again, there is no $\varepsilon(0)$ issues.

5.3.4 Fermions & Many-Body Theory

As already mentioned above, a system of N states of energies $\hbar\omega_i$, $i = 1, \ldots, N$ can be described by the Grassmann algebra $\mathfrak{U}$ of analytic functions with generators $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_N\}^T$.

$$\psi(\boldsymbol{\theta}) = \sum_{m_1=0}^{1} \ldots \sum_{m_N=0}^{1} c_{m_1 \ldots m_N} \theta_1^{m_1} \ldots \theta_N^{m_N} \qquad \forall\ \psi \in \mathfrak{U}$$

$$= \sum_{\boldsymbol{m}} c_{\boldsymbol{m}} \theta_1^{m_1} \ldots \theta_N^{m_N} \qquad \boldsymbol{m} = (m_1, \ldots, m_N) \tag{5.74a}$$

The dimension of (or number of basis functions in) $\mathfrak{U}$ is therefore 2^N.

Note:

1. $\psi(\boldsymbol{\theta})$ represents a mixture of 0- to N-particle function.
2. Pauli' exclusion principle is automatically realized.
2. $\boldsymbol{\theta}^+ = \left\{\overline{\theta}_1, \ldots, \overline{\theta}_N\right\}$ is a member of the dual space $\mathfrak{U}^+$ introduced by the inner product.

Another way to expand $\psi(\boldsymbol{\theta})$ which emphasizes its anti-symmetry is as follows.

Let P_n^N be the set of sets of indices obtained by picking n indices out of the set $\{1, \ldots, N\}$. Then

$$\psi(\boldsymbol{\theta}) = \sum_{n=0}^{N} \frac{1}{n!} \sum_{\boldsymbol{k} \in P_n^N} \psi_{\boldsymbol{k}} \theta_{k_1} \ldots \theta_{k_n} \qquad \boldsymbol{k} = (k_1, \ldots, k_n) \tag{5.74b}$$

where $\psi_{\boldsymbol{k}}$ is antisymmetric in its indices & there are $\frac{N!}{(N-n)!}$ members in P_n^N.

For the example of $N = 2$, we have

For $n = 0$, $\quad \frac{N!}{(N-n)!} = 1, \qquad P_0^N = \{\boldsymbol{k}\} \qquad \boldsymbol{k} = 0$

For $n = 1$, $\quad \frac{N!}{(N-n)!} = 2, \qquad P_1^N = \{\boldsymbol{k}_1, \boldsymbol{k}_2\}\ \boldsymbol{k}_1 = 1 \qquad \boldsymbol{k}_2 = 2$

For $n = 2$, $\frac{N!}{(N-n)!} = 2$, $P_2^N = \{\boldsymbol{k}_1, \boldsymbol{k}_2\}$ $\boldsymbol{k}_1 = (1, 2)$ $\boldsymbol{k}_2 = (2, 1)$

$$\rightarrow \quad \psi(\boldsymbol{\theta}) = \psi_0 + \psi_1\,\theta_1 + \psi_2\,\theta_2 + \frac{1}{2}(\psi_{12}\,\theta_1\,\theta_2 + \psi_{21}\,\theta_2\,\theta_1) \quad \text{with} \quad \psi_{12} = -\psi_{21}$$

The free hamiltonian is obviously

$$H_0 = \sum_i \hbar\,\omega_i\,\theta_i\,\frac{\partial}{\partial\,\theta_i}$$

If only n states labeled by $\boldsymbol{k} = (k_1, \ldots, k_n)$ are occupied, the energy of the system is

$$E_{\boldsymbol{k}} = \sum_{j=1}^{n} \hbar\,\omega_{k_j}$$

We shall interpret $\theta_{k_1} \ldots \theta_{k_n}$ as representing an n-particle state in which particle j occupies state k_j. The antisymmetrized eigenstate $\psi_{\boldsymbol{k}}$ with energy $E_{\boldsymbol{k}}$ is therefore

$$\psi_{\boldsymbol{k}} = \sum_P c_{P\,\boldsymbol{k}}\,\theta_{k_{P(1)}} \ldots \theta_{k_{P(n)}}$$

where P is a permutation of $(1, \ldots, n)$ so that

$$P\,\boldsymbol{k} = (k_{P(1)}, \ldots, k_{P(n)})$$

and $c_{P\,\boldsymbol{k}}$ is antisymmetric in its indices.

For the example of 2 particles occupying states 3 & 5, we have

$$E = \hbar\,\omega_3 + \hbar\,\omega_5$$
$$\psi = \psi_{35}\,\theta_3\,\theta_5 + \psi_{53}\,\theta_5\,\theta_3 \quad \text{with} \quad \psi_{35} = -\psi_{53}$$
$$= 2\,\psi_{35}\,\theta_3\,\theta_5$$
$$= c_{00101}\,\theta_3\,\theta_5$$

Thus, explicitly anti-symmetric eigenstates are pointless in Grassmann algebra & eq(5.74a) is preferred over eq(5.74b).

As already hinted in the derivation of eq(5.53a), the generalization to a many-state system can be achieved by replacing θ with $\boldsymbol{\theta}$. For example, eq(5.65) becomes

$$S_0(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\;\; \boldsymbol{\theta}^+(t)\left[\dot{\boldsymbol{\theta}}(t) - \omega\,\boldsymbol{\theta}(t)\right] \tag{5.74}$$

with anti-periodic B.C.

$$\boldsymbol{\theta}(\hbar\beta/2) = -\boldsymbol{\theta}(-\hbar\beta/2) \quad \& \quad \boldsymbol{\theta}^+(\hbar\beta/2) = -\boldsymbol{\theta}^+(-\hbar\beta/2)$$

Interactions

Interaction between particles can be introduced as cross-terms like $\overline{\theta}_i\,\theta_j$ in the hamiltonian H. If H is in normal ordered form, then by eq(5.48)

$$\langle\,\boldsymbol{\theta}\mid H \mid \boldsymbol{\theta}'\,\rangle = H\left(\boldsymbol{\theta}, \frac{\partial}{\partial\,\boldsymbol{\theta}}\right)\langle\,\boldsymbol{\theta}\mid\boldsymbol{\theta}'\,\rangle = H(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}')\,e^{-\boldsymbol{\theta}'^{+}\boldsymbol{\theta}} \tag{5.75a}$$

Eq(5.57) then takes the form

$$\langle\,\boldsymbol{\theta}\mid U(\varepsilon)\mid\boldsymbol{\theta}'\,\rangle = \exp\left[-\boldsymbol{\theta}'^{+}\boldsymbol{\theta} - \varepsilon\,H(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}') + O\left(\varepsilon^2\right)\right] \tag{5.75}$$

For finite times, eq(5.61) generalizes to

$$\langle\,\boldsymbol{\theta}''\mid U(t'', t')\mid\boldsymbol{\theta}'\,\rangle = \int_{\boldsymbol{\theta}(t')=\boldsymbol{\theta}'}^{\boldsymbol{\theta}(t'')=\boldsymbol{\theta}''}\left[d\,\boldsymbol{\theta}(t)\,d\,\overline{\boldsymbol{\theta}}(t)\right]e^{-S(\boldsymbol{\theta},\overline{\boldsymbol{\theta}})} \tag{5.76}$$

with the counterpart of eq(5.62) being

$$S(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}) = \boldsymbol{\theta}^{+}(t')\,\boldsymbol{\theta}(t') + \int_{t'}^{t''} d\,t\;\; \boldsymbol{\theta}^{+}(t)\left[\dot{\boldsymbol{\theta}}(t) - H(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}})\right] \tag{5.77}$$

Remark

In the 2nd quantization scheme, one set

$$\{a_i, a_j^{+}\} = \delta_{i\,j} \qquad \{a_i^{+}, a_j^{+}\} = \{a_i, a_j\} = 0 \tag{5.78}$$

where

$$a_i^{+} \mapsto \theta_i \qquad a_i \mapsto \frac{\partial}{\partial\,\theta_i}$$

so that

$$H = H(\boldsymbol{a}^{+}, \boldsymbol{a}) \mapsto H\left(\boldsymbol{\theta}, \frac{\partial}{\partial\,\boldsymbol{\theta}}\right)$$