5.5. The Bose Gas. Functional Integrals

5.5.1. Fock's space and hamiltonian

Let $\mathcal{H}_n$ be the Hilbert space of n-particle boson states in d space dimensions &

$$H_n = T_n + \mathcal{V}_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n) \tag{5.82}$$

with

$$T_n = -\frac{\hbar^2}{2\,m}\sum_{i=1}^{n} \nabla_i^2 \qquad\qquad \nabla_i \equiv \frac{\partial}{\partial\, \boldsymbol{x}_i} \tag{5.83}$$

and $\mathcal{V}_n$ is a symmetric function of its arguments. In particular, we consider only

$$\mathcal{V}_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n) = \sum_{i=1}^{n} V_1(\boldsymbol{x}_i) + \sum_{i<j\le n} V_2(\boldsymbol{x}_i\,, \boldsymbol{x}_j) \tag{5.84}$$

where

$$V_2(\boldsymbol{x}, \boldsymbol{y}) = V_2(\boldsymbol{y}, \boldsymbol{x})$$

In the absence of internal degrees of freedom, the n-particle wave function is a complex function $\psi_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n)$ symmetric in its arguments. $\mathcal{H}_n$ is therefore the space of all square integrable symmetric functions.

The direct sum $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$ is called a Fock space $\mathcal{F}$. Operators in $\mathcal{F}$ will be denoted by bold-face letters, e.g.

$$\boldsymbol{H} = \boldsymbol{T} + \boldsymbol{V}_1 + \boldsymbol{V}_2 \tag{5.85}$$

describes a hamiltonian of arbitrary number of bosons.

Generating Functional of Wave Functions & Hamiltonian

Using eq(1.97) of §1.9, we define the generating functional

$$\begin{aligned}\boldsymbol{\Psi}(\varphi) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \ldots d^d x_n\, \psi_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n)\, \varphi(\boldsymbol{x}_1) \ldots \varphi(\boldsymbol{x}_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} \left[d^d x_i\, \varphi(\boldsymbol{x}_i) \right] \psi_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n)\end{aligned} \tag{5.86}$$

where $\psi_n(\boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_n)$ is symmetric in its arguments, i.e., ψ_n is a linear combination of $n!$ terms of equal weight. After the integrations, each of these terms gives the same result so that the $\frac{1}{n!}$ factor is cancelled out.

As described in §1.9, all operators in $\mathcal{F}$ can be written as functional derivatives of $\boldsymbol{\Psi}$.

Using eq(1.100), we have

$$\frac{\delta}{\delta\,\varphi(\boldsymbol{x})}\boldsymbol{\Psi}(\varphi) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n-1} \left[d^d x_i\, \varphi(\boldsymbol{x}_i) \right] \psi_n(\boldsymbol{x}, \boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_{n-1}) \tag{5.86a}$$

$$\begin{aligned}\therefore \qquad \mathcal{I} &= \int d^d x\, \varphi(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2 \frac{\delta}{\delta\,\varphi(\boldsymbol{x})} \boldsymbol{\Psi}(\varphi) \\ &= \int d^d x\, \varphi(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n-1} \left[d^d x_i\, \varphi(\boldsymbol{x}_i) \right] \psi_n(\boldsymbol{x}, \boldsymbol{x}_1, \ \ldots, \ \boldsymbol{x}_{n-1})\end{aligned}$$

Renaming $\boldsymbol{x} = \boldsymbol{x}_n$, we have (order of arguments immaterial)

$$\mathcal{I} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int \prod_{i=1}^{n} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] \nabla_n^2 \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] \sum_{j=1}^{n} \nabla_j^2 \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \qquad (\ \psi_n \text{ symmetric. })$$

$$\therefore \qquad -\frac{\hbar^2}{2m} \mathcal{I} = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] T_n\, \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$= \int d^d \boldsymbol{x}\, \varphi(\boldsymbol{x})\, T_1\, \psi_1(\boldsymbol{x}) + \frac{1}{2} \int d^d \boldsymbol{x}_1\, d^d \boldsymbol{x}_2\, \varphi(\boldsymbol{x}_1)\, \varphi(\boldsymbol{x}_2)\, T_2\, \psi_2(\boldsymbol{x}_1, \boldsymbol{x}_2) + \ldots$$

$$\equiv \hat{\boldsymbol{T}}\, \boldsymbol{\Psi}(\varphi)$$

where we’ve use the overhead “∧” to denote an operator with functional derivatives.

For example, since we can always write ψ_2 as

$$\psi_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = \phi_1(\boldsymbol{x}_1)\, \phi_2(\boldsymbol{x}_2) + \phi_1(\boldsymbol{x}_2)\, \phi_2(\boldsymbol{x}_1)$$

we have

$$T_2\, \psi_2(\boldsymbol{x}_1, \boldsymbol{x}_2) = [\, T(\boldsymbol{x}_1) + T(\boldsymbol{x}_2)\,][\, \phi_1(\boldsymbol{x}_1)\, \phi_2(\boldsymbol{x}_2) + \phi_1(\boldsymbol{x}_2)\, \phi_2(\boldsymbol{x}_1)\,]$$

$$= T\, \phi_1(\boldsymbol{x}_1)\, \phi_2(\boldsymbol{x}_2) + \phi_1(\boldsymbol{x}_2)\, T\, \phi_2(\boldsymbol{x}_1) + \phi_1(\boldsymbol{x}_1)\, T\, \phi_2(\boldsymbol{x}_2) + T\, \phi_1(\boldsymbol{x}_2)\, \phi_2(\boldsymbol{x}_1)$$

After the integration over $\boldsymbol{x}_1$ & $\boldsymbol{x}_2$, the 1st & 4th terms give the same result, so do the 2nd & 3rd terms .

Thus, $T_n\, \psi_n$ is just a sum of n terms each of which T operates on only one $\boldsymbol{x}_j$.

Furthermore, the factor $\frac{1}{n!}$ is cancelled if ψ_n is written out explicitly.

$$\hat{\boldsymbol{T}}\, \boldsymbol{\Psi}(\varphi) = -\frac{\hbar^2}{2m} \int d^d x\, \varphi(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2 \frac{\delta}{\delta \varphi(\boldsymbol{x})} \boldsymbol{\Psi}(\varphi) \qquad (5.87)$$

Since $\boldsymbol{\Psi}$, via ψ_n, is arbitrary, we have

$$\hat{\boldsymbol{T}} = -\frac{\hbar^2}{2m} \int d^d x\, \varphi(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2 \frac{\delta}{\delta \varphi(\boldsymbol{x})} \qquad (5.87a)$$

Replacing $-\frac{\hbar^2}{2m} \nabla_j^2$ with $V_1(\boldsymbol{x}_j)$, we get

$$\hat{\boldsymbol{V}}_1 = \int d^d x\, \varphi(\boldsymbol{x})\, V_1(\boldsymbol{x}) \frac{\delta}{\delta \varphi(\boldsymbol{x})} \qquad (5.87b)$$

From eq(5.86a), we have

$$\frac{\delta}{\delta \varphi(\boldsymbol{x})} \frac{\delta}{\delta \varphi(\boldsymbol{y})} \boldsymbol{\Psi}(\varphi) = \sum_{n=1}^{\infty} \frac{1}{(n-2)!} \int \prod_{i=1}^{n-2} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] \psi_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-2})$$

so that

$$\mathcal{I} = \frac{1}{2} \int d^d x\, d^d y\, \varphi(\boldsymbol{x})\, \varphi(\boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y}) \frac{\delta}{\delta \varphi(\boldsymbol{x})} \frac{\delta}{\delta \varphi(\boldsymbol{y})} \boldsymbol{\Psi}(\varphi)$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \int \prod_{i=1}^{n} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] V_2(\boldsymbol{x}_{n-1}, \boldsymbol{x}_n)\, \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [\, d^d x_i\, \varphi(\boldsymbol{x}_i)\,] \sum_{j \neq k} V_2(\boldsymbol{x}_j, \boldsymbol{x}_k)\, \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

$$\equiv \hat{\boldsymbol{V}}_2\, \boldsymbol{\Psi}(\varphi)$$

$$\therefore \qquad \hat{V}_2 = \frac{1}{2}\int d^d x\, d^d y\, \varphi(\mathbf{y})\, \varphi(\mathbf{x})\, V_2(\mathbf{x},\mathbf{y}) \frac{\delta}{\delta\varphi(\mathbf{y})} \frac{\delta}{\delta\varphi(\mathbf{x})} \tag{5.88}$$

The density operator for an n-particle system

$$\rho_n(\mathbf{x}) = \sum_{i=1}^{n} \delta(\mathbf{x}-\mathbf{x}_i)$$

gives the density of particles at $\mathbf{x}$. The corresponding number operator is therefore

$$N_n = \int d^d x\, \rho_n(\mathbf{x}) = n$$

Replacing $V_1(\mathbf{x}_j)$ with $\delta(\mathbf{x}-\mathbf{x}_j)$ in eq(5.87a), we get

$$\begin{aligned}
\mathcal{I}(\mathbf{y}) &= \int d^d x\, \varphi(\mathbf{x})\, \delta(\mathbf{y}-\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})} \mathbf{\Psi}(\varphi) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [d^d x_i\, \varphi(\mathbf{x}_i)] \sum_{j=1}^{n} \delta(\mathbf{y}-\mathbf{x})\, \psi_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} [d^d x_i\, \varphi(\mathbf{x}_i)]\, \rho_n(\mathbf{y})\, \psi_n(\mathbf{x}_1, \ldots, \mathbf{x}_n) \\
&\equiv \hat{\boldsymbol{\rho}}(\mathbf{y})\, \mathbf{\Psi}(\varphi)
\end{aligned}$$

Thus, the Fock space density operator $\boldsymbol{\rho}(\mathbf{y})$ is given by

$$\hat{\boldsymbol{\rho}}(\mathbf{y}) = \int d^d x\, \varphi(\mathbf{x})\, \delta(\mathbf{y}-\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})} \tag{5.89a}$$

The number operator in $\mathcal{F}$

$$\hat{\mathbf{N}} = \int d^d y\, \hat{\boldsymbol{\rho}}(\mathbf{y})$$

therefore becomes

$$\hat{\mathbf{N}} = \int d^d x\, \varphi(\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})} \tag{5.89}$$

Proof of [N , H] = 0

$$[\hat{\mathbf{N}}, \hat{\mathbf{T}}]\mathbf{\Psi}(\varphi) = -\frac{\hbar^2}{2m}\int d^d x\, d^d y \left[\varphi(\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})}, \varphi(\mathbf{y}) \nabla_y^2 \frac{\delta}{\delta\varphi(\mathbf{y})}\right]\mathbf{\Psi}(\varphi)$$

$$\begin{aligned}
&\left[\varphi(\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})}, \varphi(\mathbf{y}) \nabla_y^2 \frac{\delta}{\delta\varphi(\mathbf{y})}\right]\mathbf{\Psi}(\varphi) \\
&= \varphi(\mathbf{x}) \frac{\delta}{\delta\varphi(\mathbf{x})}\left(\varphi(\mathbf{y}) \nabla_y^2 \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{y})}\right) - \varphi(\mathbf{y}) \nabla_y^2 \left[\frac{\delta}{\delta\varphi(\mathbf{y})}\left(\varphi(\mathbf{x}) \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})}\right)\right] \\
&= \varphi(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}) \nabla_y^2 \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{y})} + \varphi(\mathbf{x})\, \varphi(\mathbf{y}) \nabla_y^2 \frac{\delta^2\mathbf{\Psi}}{\delta\varphi(\mathbf{x})\, \delta\varphi(\mathbf{y})} \\
&\quad -\varphi(\mathbf{y}) \nabla_y^2 \left(\delta(\mathbf{x}-\mathbf{y}) \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})}\right) - \varphi(\mathbf{y})\, \varphi(\mathbf{x}) \nabla_y^2 \frac{\delta^2\mathbf{\Psi}}{\delta\varphi(\mathbf{x})\, \delta\varphi(\mathbf{y})} \\
&= \varphi(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{y}) \nabla_y^2 \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})} - \varphi(\mathbf{y}) \nabla_y^2 \left(\delta(\mathbf{x}-\mathbf{y}) \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})}\right)
\end{aligned}$$

Using

$$\nabla_y^2\left(\delta(\mathbf{x}-\mathbf{y}) \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})}\right) = \nabla_y \cdot \left\{[\nabla_y \delta(\mathbf{x}-\mathbf{y})] \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})} + \delta(\mathbf{x}-\mathbf{y}) \nabla_y \frac{\delta\mathbf{\Psi}}{\delta\varphi(\mathbf{x})}\right\}$$

$$= \left[\nabla_{\boldsymbol{y}}^2 \delta(\boldsymbol{x}-\boldsymbol{y}) \right] \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})} + 2 \left[\nabla_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y}) \right] \cdot \nabla_{\boldsymbol{y}} \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})} + \delta(\boldsymbol{x}-\boldsymbol{y}) \nabla_{\boldsymbol{y}}^2 \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})}$$

we have

$$\left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ \varphi(\boldsymbol{y}) \nabla_{\boldsymbol{y}}^2 \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right] \boldsymbol{\Psi}(\varphi)$$

$$= -\varphi(\boldsymbol{y}) \left[\nabla_{\boldsymbol{y}}^2 \delta(\boldsymbol{x}-\boldsymbol{y}) \right] \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})} - 2\, \varphi(\boldsymbol{y}) \left[\nabla_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y}) \right] \cdot \nabla_{\boldsymbol{y}} \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})}$$

Using

$$\int d^d y\, \nabla \cdot \boldsymbol{f} = \oint d\, \boldsymbol{S} \cdot \boldsymbol{f} = 0 \qquad \boldsymbol{S} = \text{surface enclosing volume of integration}$$

we have

$$\int d^d y\, \nabla_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{f}(\boldsymbol{y}) = \int d^d y \left\{ \nabla_{\boldsymbol{y}} \cdot [\, \delta(\boldsymbol{x}-\boldsymbol{y})\, \boldsymbol{f}(\boldsymbol{y})\,] - \delta(\boldsymbol{x}-\boldsymbol{y}) \nabla_{\boldsymbol{y}} \cdot \boldsymbol{f}(\boldsymbol{y}) \right\}$$

$$= -\nabla_{\boldsymbol{x}} \cdot \boldsymbol{f}(\boldsymbol{x})$$

$$\int d^d y\, \nabla_{\boldsymbol{y}}^2 \delta(\boldsymbol{x}-\boldsymbol{y})\, g(\boldsymbol{y}) = \int d^d y \left\{ \nabla_{\boldsymbol{y}} \cdot \left[g(\boldsymbol{y}) \nabla_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y}) \right] - \nabla_{\boldsymbol{y}}\, g(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \delta(\boldsymbol{x}-\boldsymbol{y}) \right\}$$

$$= \nabla_{\boldsymbol{x}}^2 g(\boldsymbol{x})$$

we have

$$\left[\hat{\boldsymbol{N}},\ \hat{\boldsymbol{T}} \right] \boldsymbol{\Psi}(\varphi) = -\frac{\hbar^2}{2\, m} \int d^d x \left\{ -\nabla_{\boldsymbol{x}}^2 \left[\varphi(\boldsymbol{x}) \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})} \right] + 2\, \nabla_{\boldsymbol{x}} \cdot \left[\varphi(\boldsymbol{x}) \nabla_{\boldsymbol{x}} \frac{\delta \boldsymbol{\Psi}}{\delta\, \varphi(\boldsymbol{x})} \right] \right\}$$

$$= 0$$

$$\left[\hat{\boldsymbol{N}},\ \hat{\boldsymbol{V}}_1 \right] \boldsymbol{\Psi}(\varphi) = \int d^d x\, d^d y \left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ \varphi(\boldsymbol{y})\, V_1(\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right] \boldsymbol{\Psi}(\varphi)$$

Using

$$\left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ \varphi(\boldsymbol{y})\, V_1(\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right] = V_1(\boldsymbol{y}) \left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ \varphi(\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right]$$

$$= V_1(\boldsymbol{y}) \left\{ \varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})} \left[\varphi(\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right] - \varphi(\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})} \right] \right\}$$

$$= V_1(\boldsymbol{y}) \left\{ \varphi(\boldsymbol{x})\, \delta(\boldsymbol{x}-\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{y})} \right] + \varphi(\boldsymbol{x})\, \varphi(\boldsymbol{y}) \frac{\delta^2}{\delta\, \varphi(\boldsymbol{x})\, \delta\, \varphi(\boldsymbol{y})}$$

$$-\varphi(\boldsymbol{y})\, \delta(\boldsymbol{x}-\boldsymbol{y}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})} \Big] - \varphi(\boldsymbol{y})\, \varphi(\boldsymbol{x}) \frac{\delta^2}{\delta\, \varphi(\boldsymbol{y})\, \delta\, \varphi(\boldsymbol{x})} \Big\}$$

$$= 0$$

we have

$$\left[\hat{\boldsymbol{N}},\ \hat{\boldsymbol{V}}_1 \right] = 0$$

Similarly,

$$\left[\hat{\boldsymbol{N}},\ \hat{\boldsymbol{V}}_2 \right] = \frac{1}{2} \int d^d x\ d^d y\, d^d z \left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ V_2(\boldsymbol{y}, \boldsymbol{z})\, \varphi(\boldsymbol{y})\, \varphi(\boldsymbol{z}) \frac{\delta^2}{\delta\, \varphi(\boldsymbol{y})\, \delta\, \varphi(\boldsymbol{z})} \right]$$

$$\left[\varphi(\boldsymbol{x}) \frac{\delta}{\delta\, \varphi(\boldsymbol{x})},\ \varphi(\boldsymbol{y})\, \varphi(\boldsymbol{z}) \frac{\delta^2}{\delta\, \varphi(\boldsymbol{y})\, \delta\, \varphi(\boldsymbol{z})} \right]$$

$$= \varphi(\mathbf{x}) \frac{\delta}{\delta \varphi(\mathbf{x})}\left[\varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}\right]$$

$$-\varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}\left[\varphi(\mathbf{x}) \frac{\delta}{\delta \varphi(\mathbf{x})}\right]$$

$$= \varphi(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})} + \varphi(\mathbf{x}) \varphi(\mathbf{y}) \delta(\mathbf{x}-\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}$$

$$+\varphi(\mathbf{x}) \varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^3}{\delta \varphi(\mathbf{x}) \delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}$$

$$-\varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta}{\delta \varphi(\mathbf{y})}\left[\delta(\mathbf{x}-\mathbf{z}) \frac{\delta}{\delta \varphi(\mathbf{x})} + \varphi(\mathbf{x}) \frac{\delta^2}{\delta \varphi(\mathbf{z}) \delta \varphi(\mathbf{x})}\right]$$

$$= \varphi(\mathbf{x}) \delta(\mathbf{x}-\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})} + \varphi(\mathbf{x}) \varphi(\mathbf{y}) \delta(\mathbf{x}-\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}$$

$$-\varphi(\mathbf{y}) \varphi(\mathbf{z}) \delta(\mathbf{x}-\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{x})} - \varphi(\mathbf{y}) \varphi(\mathbf{z}) \delta(\mathbf{x}-\mathbf{y}) \frac{\delta^2}{\delta \varphi(\mathbf{z}) \delta \varphi(\mathbf{x})}$$

$$\therefore \qquad \left[\hat{\mathbf{N}}, \hat{\mathbf{V}}_2\right] = \frac{1}{2}\int d^d y\, d^d z\, V_2(\mathbf{y}, \mathbf{z})\left\{\varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}\right.$$

$$+\varphi(\mathbf{z}) \varphi(\mathbf{y}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})} - \varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{y}) \delta \varphi(\mathbf{z})}$$

$$\left.-\varphi(\mathbf{y}) \varphi(\mathbf{z}) \frac{\delta^2}{\delta \varphi(\mathbf{z}) \delta \varphi(\mathbf{y})}\right\}$$

$$= 0$$

Hence,

$$\left[\hat{\mathbf{N}}, \hat{\mathbf{H}}\right] = 0 \qquad (5.89a)$$

→ Eigenstates of $\mathbf{H}$ are also eigenstates of $\mathbf{H} - \mu \mathbf{N}$.

Analytic Functions of *n* Variables

Here, we generalize the results of §5.1 to the case of multi-variables.

Let

$$\mathbf{z} = (z_1, \ldots, z_n)^T \qquad \mathbf{z}^+ = (\bar{z}_1, \ldots, \bar{z}_n)$$

then eq(5.7) generalizes to

$$(g, f) = \int \prod_{i=1}^{n} \frac{d\bar{z}_i\, dz_i}{2\pi i} e^{-\sum_{j=1}^{n} z_i \bar{z}_i} \overline{g(\mathbf{z})} f(\mathbf{z})$$

$$\equiv \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^+ \mathbf{z}} \overline{g(\mathbf{z})} f(\mathbf{z}) \qquad (5.90a)$$

$$= \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^+ \mathbf{z}} \langle g \mid \mathbf{z}\rangle\langle \mathbf{z} \mid f\rangle$$

$$\rightarrow \qquad \int \prod_i \frac{d\bar{z}_i\, dz_i}{2\pi i} e^{-\sum_i z_i \bar{z}_i} \mid z_1 \ldots z_n\rangle\langle z_1 \ldots z_n \mid = I$$

or $$\int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^{+}\mathbf{z}} \mid \mathbf{z}\rangle\langle \mathbf{z} \mid = I \tag{5.90b}$$

The vector space $\mathcal{A}$ of analytic functions is spanned by the orthonormal basis

$$\langle \mathbf{z} \mid m_1, \ldots, m_n \rangle \equiv \langle \mathbf{z} \mid \mathbf{m} \rangle = \frac{z_1^{m_1}}{\sqrt{m_1!}} \cdots \frac{z_n^{m_n}}{\sqrt{m_n}}$$

where

$$\sum_{\mathbf{m}} \mid \mathbf{m}\rangle\langle \mathbf{m} \mid = I \tag{5.90c}$$

$$\rightarrow \quad \sum_{\mathbf{m}} \langle \mathbf{z} \mid \mathbf{m}\rangle\langle \mathbf{m} \mid \mathbf{z}'\rangle = \langle \mathbf{z} \mid \mathbf{z}'\rangle = e^{\mathbf{z}'^{+}\mathbf{z}} = \exp\left(\sum_{i=1}^{n} \bar{z}_i' z_i\right) \tag{5.90d}$$

Similarly,

$$f(\mathbf{z}') = \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^{+}(\mathbf{z}-\mathbf{z}')} f(\mathbf{z}) \tag{5.90e}$$

$$\delta(\mathbf{z}-\mathbf{z}') = \int \frac{d\bar{\mathbf{z}}}{2\pi i} e^{-\mathbf{z}^{+}(\mathbf{z}-\mathbf{z}')} \tag{5.90f}$$

$$\langle \mathbf{z} \mid O \mid \mathbf{z}'\rangle = O\left(\mathbf{z}, \frac{\partial}{\partial \mathbf{z}}\right)\langle \mathbf{z} \mid \mathbf{z}'\rangle$$

$$= O\left(\mathbf{z}, \frac{\partial}{\partial \mathbf{z}}\right) e^{\mathbf{z}'^{+}\mathbf{z}} \tag{5.90g}$$

$$\langle \mathbf{z} \mid \mathcal{N} \mid \mathbf{z}'\rangle = \mathcal{N}(\mathbf{z}, \bar{\mathbf{z}})\, e^{\mathbf{z}'^{+}\mathbf{z}} \tag{5.90h}$$

$$\langle \mathbf{z}' \mid O_2 O_1 \mid \mathbf{z}''\rangle = \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^{+}\mathbf{z}} \langle \mathbf{z}' \mid O_2 \mid \mathbf{z}\rangle\langle \mathbf{z} \mid O_1 \mid \mathbf{z}''\rangle \tag{5.90i}$$

$$\operatorname{tr} O = \sum_{\mathbf{m}} \langle \mathbf{m} \mid O \mid \mathbf{m}\rangle$$

$$= \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} e^{-\mathbf{z}^{+}\mathbf{z}} \langle \mathbf{z} \mid O \mid \mathbf{z}\rangle \tag{5.90j}$$

$$H_0 = \sum_{i=1}^{n} \hbar\, \omega_i\, z_i \frac{\partial}{\partial z_i} \tag{5.90k}$$

$$\langle \mathbf{z} \mid H_0 \mid \mathbf{z}'\rangle = H_0(\mathbf{z}, \bar{\mathbf{z}}')\, e^{\mathbf{z}'^{+}\mathbf{z}} \tag{5.90l}$$

$$H_0\, \psi_{\mathbf{m}} = E_{\mathbf{m}}\, \psi_m \qquad E_{\mathbf{m}} = \sum_{i=1}^{n} m_i\, \hbar\, \omega_i \tag{5.90m}$$

$$\psi_{\mathbf{m}}(\mathbf{z}) = \langle \mathbf{m} \mid \mathbf{z}\rangle \tag{5.90n}$$

$$U_0(t)\, f(\mathbf{z}) = f\left(e^{-\omega t}\, \mathbf{z}\right) \tag{5.90o}$$

$$\langle \mathbf{z} \mid U_0(t) \mid \mathbf{z}'\rangle = \exp\left(e^{-\omega t}\, \mathbf{z}'^{+}\mathbf{z}\right) \tag{5.90p}$$

$$O = \sum_{\mathbf{m},\mathbf{n}} \mid \mathbf{m}\rangle\langle \mathbf{m} \mid O \mid \mathbf{n}\rangle\langle \mathbf{n} \mid$$

$$= \int \frac{d\bar{\mathbf{z}}\, d\mathbf{z}}{2\pi i} \frac{d\bar{\mathbf{z}}'\, d\mathbf{z}'}{2\pi i} e^{-\mathbf{z}^{+}\mathbf{z}-\mathbf{z}'^{+}\mathbf{z}'} \mid \mathbf{z}\rangle\langle \mathbf{z}' \mid \tag{5.90q}$$

$$\times \sum_{\mathbf{m},\mathbf{n}} O_{\mathbf{m}\,\mathbf{n}} \frac{z_1^{m_1}}{\sqrt{m_1!}} \cdots \frac{z_n^{m_n}}{\sqrt{m_n!}} \frac{\bar{z}_1'^{n_1}}{\sqrt{n_1!}} \cdots \frac{\bar{z}_n'^{n_n}}{\sqrt{n_n!}}$$

5.5.2. Functional Integral

Warning: As in §§5.1-2, we'll work with matrix elements of the type $\langle \overline{\varphi} \mid O \mid \overline{\varphi}' \rangle$ instead of Zinn-Justin's $\langle \varphi \mid O \mid \overline{\varphi} \rangle$. However, such differences will no longer be emphasized with special equation labels.

In adapting the holomorphic results to the present case, we use eq(5.4) to establish the correspondence

$$z \sim a^+ \sim \overline{\varphi} \qquad \& \qquad \overline{z} \sim a \sim \varphi \tag{5.90r}$$

Treating $\overline{\varphi}(\boldsymbol{x})$ as the dynamic variable, with $\boldsymbol{x}$ standing for an d-D continuous index, the completeness relation eq(5.90b) becomes

$$\int [d\,\overline{\varphi}(\boldsymbol{x})\, d\,\varphi(\boldsymbol{x})] \exp\left[-\int d^d x\, \varphi(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{x})\right] \mid \overline{\varphi}(\boldsymbol{x}) \rangle \langle \overline{\varphi}(\boldsymbol{x}) \mid = I$$

or simply

$$\int [d\,\overline{\varphi}\, d\,\varphi] \exp\left(-\int d\boldsymbol{x}\, \varphi\, \overline{\varphi}\right) \mid \overline{\varphi} \rangle \langle \overline{\varphi} \mid = I \tag{5.90s}$$

where

$$[d\,\overline{\varphi}\, d\,\varphi] = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{d\,\overline{\varphi}(\boldsymbol{x}_k)\, d\,\varphi(\boldsymbol{x}_k)}{2\pi i}$$

Eq(5.90d) becomes

$$\langle \overline{\varphi} \mid \overline{\varphi}' \rangle = \exp\left(\int d\,\boldsymbol{x}\, \overline{\varphi}\, \varphi'\right) \tag{5.90t}$$

The generalization of eq(5.90g) is

$$\langle \overline{\varphi} \mid O \mid \overline{\varphi}' \rangle = \hat{O}\left(\overline{\varphi}, \frac{\delta}{\delta\,\overline{\varphi}}\right) \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \tag{5.90u}$$

Using eq(5.87a), we have

$$\begin{aligned}
\hat{\boldsymbol{T}} \langle \overline{\varphi} \mid \overline{\varphi}' \rangle &= -\frac{\hbar^2}{2m} \int d\boldsymbol{x}\, \overline{\varphi}(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2 \frac{\delta}{\delta\,\overline{\varphi}(\boldsymbol{x})} \exp\left(\int d\,\boldsymbol{x}'\, \overline{\varphi}(\boldsymbol{x}')\, \varphi'(\boldsymbol{x}')\right) \\
&= -\frac{\hbar^2}{2m} \left(\int d\boldsymbol{x}\, \overline{\varphi}(\boldsymbol{x})\, \nabla_{\boldsymbol{x}}^2\, \varphi'(\boldsymbol{x})\right) \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \\
&= \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \int d\boldsymbol{x}\, \overline{\varphi}(\boldsymbol{x}) \left(-\frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2\right) \varphi'(\boldsymbol{x})
\end{aligned} \tag{5.90v}$$

Similarly, eqs(5.87b & 88a) give

$$\hat{\boldsymbol{V}}_1 \langle \overline{\varphi} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \int d\boldsymbol{x}\, \overline{\varphi}(\boldsymbol{x})\, V_1(\boldsymbol{x})\, \varphi'(\boldsymbol{x}) \tag{5.90w}$$

$$\hat{\boldsymbol{V}}_2 \langle \overline{\varphi} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \frac{1}{2} \int d\boldsymbol{x}\, d\,\boldsymbol{y}\;\; \overline{\varphi}(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi'(\boldsymbol{y})\, \varphi'(\boldsymbol{x}) \tag{5.90x}$$

Hence,

$$\begin{aligned}
\langle \overline{\varphi} \mid \boldsymbol{H} \mid \overline{\varphi}' \rangle &= \hat{\boldsymbol{H}} \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \\
&= \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \left\{ \int d\boldsymbol{x}\;\; \overline{\varphi}(\boldsymbol{x}) \left[-\frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x})\right] \varphi'(\boldsymbol{x}) \right. \\
&\qquad \left. + \frac{1}{2} \int d\boldsymbol{x}\, d\boldsymbol{y}\, \overline{\varphi}(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi'(\boldsymbol{y})\, \varphi'(\boldsymbol{x}) \right\}
\end{aligned} \tag{5.90}$$

Using eq(5.89), we have

$$\langle \overline{\varphi} \mid \boldsymbol{N} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \int d^d x\, \overline{\varphi}(\boldsymbol{x})\, \varphi'(\boldsymbol{x}) \tag{5.91}$$

Eqs(5.34p & 35a) are easily generalized to

$$\langle \overline{\varphi}'' \mid \boldsymbol{U}(t'', t') \mid \overline{\varphi}' \rangle = \left\langle \overline{\varphi}'' \,\middle|\, \exp\left[-\frac{t''-t'}{\hbar}(\boldsymbol{H}-\mu \boldsymbol{N})\right] \,\middle|\, \overline{\varphi}' \right\rangle$$
$$= \int [d\,\overline{\varphi}(t, \boldsymbol{x})\, d\,\varphi(t, \boldsymbol{x})]\, e^{-S(\varphi,\overline{\varphi})/\hbar} \tag{5.92}$$

with the B.C.

$$\overline{\varphi}(t', \boldsymbol{x}) = \overline{\varphi}'(\boldsymbol{x}) \qquad \overline{\varphi}(t'', \boldsymbol{x}) = \overline{\varphi}''(\boldsymbol{x}) \tag{5.93a}$$

After integration by part once the $\dot{\overline{\varphi}}\,\varphi$, eq(5.35) becomes

$$S(\varphi, \overline{\varphi}) = -\overline{\varphi}(t', \boldsymbol{x}')\, \varphi(t', \boldsymbol{x}') \tag{5.93}$$
$$+\int_{t'}^{t''} dt \int d\boldsymbol{x}\, \overline{\varphi}(t, \boldsymbol{x}) \left(\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi(t, \boldsymbol{x})$$
$$+\frac{1}{2} \int_{t'}^{t''} dt \int d\boldsymbol{x}\, d\boldsymbol{y}\, \overline{\varphi}(t, \boldsymbol{x})\, \overline{\varphi}(t, \boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi(t, \boldsymbol{y})\, \varphi(t, \boldsymbol{x})$$

Similarly, eqs(5.40 & 41t) generalized to

$$\mathcal{Z}(\beta) = \int [d\,\overline{\varphi}(t, \boldsymbol{x})\, d\,\varphi(t, \boldsymbol{x})]\, e^{-S(\varphi,\overline{\varphi})/\hbar} \tag{5.94}$$

with

$$S(\varphi, \overline{\varphi}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x}\, \overline{\varphi}(t, \boldsymbol{x}) \left(\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi(t, \boldsymbol{x}) \tag{5.95}$$
$$+\frac{1}{2} \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x}\, d\boldsymbol{y}\, \overline{\varphi}(t, \boldsymbol{x})\, \overline{\varphi}(t, \boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi(t, \boldsymbol{y})\, \varphi(t, \boldsymbol{x})$$

& the periodic B.C.

$$\varphi(\hbar\beta/2, \boldsymbol{x}) = \varphi(-\hbar\beta/2, \boldsymbol{x}) \qquad \overline{\varphi}(\hbar\beta/2, \boldsymbol{x}) = \overline{\varphi}(-\hbar\beta/2, \boldsymbol{x}) \tag{5.95a}$$

Remark

The particle conservation eq(5.89a) is a consequence of the global $U(1)$ gauge symmetry such that S is invariant under

$$\varphi(t, \boldsymbol{x}) \mapsto e^{i\theta}\varphi(t, \boldsymbol{x}) \qquad \overline{\varphi}(t, \boldsymbol{x}) \mapsto e^{-i\theta}\overline{\varphi}(t, \boldsymbol{x})$$

where θ is any real constant.

See Chapter 28 for the interesting case of spontaneous symmetry breaking of this symmetry during a phase transition.

5.5.3. The Gaussian Model

Free Bose gas

$$\mathcal{Z}_0(\beta) = \int [d\,\overline{\varphi}(t, \boldsymbol{x})\, d\,\varphi(t, \boldsymbol{x})]\, e^{-S_0(\varphi,\overline{\varphi})/\hbar}$$

with

$$S_0(\varphi, \overline{\varphi}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x}\, \overline{\varphi}(t, \boldsymbol{x}) \left(\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 - \mu \right) \varphi(t, \boldsymbol{x})$$

Translational invariance suggests the switch to the momentum space. Let

$$\varphi(t, \boldsymbol{x}) = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^{d/2}}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}\, \varphi^*(t, \boldsymbol{p}) \tag{5.96a}$$

$$\overline{\varphi}(t, \boldsymbol{x}) = \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^{d/2}}\, e^{-i\,\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}\, \overline{\varphi}^*(t, \boldsymbol{p})$$

Note that the power of $2\pi\hbar$ is $d/2$ instead of d as in Zinn-Justin's eq(5.96) [see "FourierTransforms.pdf" for explanation].

Using

$$\int d\,\boldsymbol{x}\;\; \overline{\varphi}(t, \boldsymbol{x})\, A(t, \boldsymbol{x}, \nabla_{\boldsymbol{x}})\, \varphi(t, \boldsymbol{x})$$

$$= \int d\boldsymbol{x} \int \frac{d\,\boldsymbol{p}'\, d\,\boldsymbol{p}}{(2\pi\hbar)^d}\; e^{-i\,\boldsymbol{p}'\cdot\boldsymbol{x}/\hbar}\, \overline{\varphi}^*(t, \boldsymbol{p}')\, A(t, \boldsymbol{x}, \nabla_{\boldsymbol{x}})\, e^{i\,\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}\, \varphi^*(t, \boldsymbol{p})$$

$$= \int d\boldsymbol{x} \int \frac{d\,\boldsymbol{p}'\, d\,\boldsymbol{p}}{(2\pi\hbar)^d}\, e^{i\,(\boldsymbol{p}-\boldsymbol{p}')\cdot\boldsymbol{x}/\hbar}\, \overline{\varphi}^*(t, \boldsymbol{p}')\, A\!\left(t, \boldsymbol{x}, \frac{i}{\hbar}\boldsymbol{p}\right) \varphi^*(t, \boldsymbol{p})$$

$$= \int d\,\boldsymbol{p}'\, d\,\boldsymbol{p}\, \delta(\boldsymbol{p} - \boldsymbol{p}')\, \overline{\varphi}^*(t, \boldsymbol{p}')\, A\!\left(t, \boldsymbol{x}, \frac{i}{\hbar}\boldsymbol{p}\right) \varphi^*(t, \boldsymbol{p})$$

$$= \int d\,\boldsymbol{p}\; \overline{\varphi}^*(t, \boldsymbol{p})\, A\!\left(t, \boldsymbol{x}, \frac{i}{\hbar}\boldsymbol{p}\right) \varphi^*(t, \boldsymbol{p})$$

we have

$$S_0\left(\varphi^*, \overline{\varphi}^*\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,\boldsymbol{p}\;\; \overline{\varphi}^*(t, \boldsymbol{p}) \left(\hbar\frac{\partial}{\partial\,t} + \frac{\boldsymbol{p}^2}{2\,m} - \mu\right) \varphi^*(t, \boldsymbol{p}) \qquad (5.97)$$

As already shown in §2.5.3, the jacobian of the transformation to k-space can be set to 1. Hence,

$$\mathcal{Z}_0(\beta) = \int [\, d\,\overline{\varphi}^*(t, \boldsymbol{p})\, d\,\varphi^*(t, \boldsymbol{p})\,]\, e^{-S_0(\varphi^*, \overline{\varphi}^*)/\hbar} \qquad (5.97a)$$

If the system is inside a hypercube of volume L^d, then $\boldsymbol{p}$ assumes only discrete values

$$\boldsymbol{p}_n = \frac{2\pi\hbar}{L}\,\boldsymbol{n} \quad = \frac{2\pi\hbar}{L}\,(n_1,\; \ldots,\, n_d) \qquad\qquad n_j \in \mathbb{Z} \qquad (5.98a)$$

Since the smallest change in n_j is $\Delta\, n_j = 1$, the discrete version of

$$\int d\boldsymbol{p}\, f(\boldsymbol{p}) = \int d\,p_1 \ldots d\,p_d\;\, f(\boldsymbol{p})$$

is

$$\sum_{\boldsymbol{n}} \left(\frac{2\pi\hbar}{L}\right)^d \Delta\, n_1 \ldots \Delta\, n_d\, f\,(\boldsymbol{p}_{\boldsymbol{n}}) = \left(\frac{2\pi\hbar}{L}\right)^d \sum_{\boldsymbol{n}} f\,(\boldsymbol{p}_{\boldsymbol{n}})$$

$$\rightarrow \qquad \sum_{\boldsymbol{p}_n} \rightarrow\; L^d \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} \qquad (5.98b)$$

Setting

$$\varphi(t, \boldsymbol{x}) = \frac{1}{L^{d/2}} \sum_{\boldsymbol{n}} e^{i\,\boldsymbol{p}_{\boldsymbol{n}}\cdot\boldsymbol{x}/\hbar}\, \varphi^*\,(t, \boldsymbol{p}_{\boldsymbol{n}})$$

we have

$$S_0\!\left(\varphi^*, \overline{\varphi}^*\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \sum_{\boldsymbol{n}}\; \overline{\varphi}^*(t, \boldsymbol{p}_{\boldsymbol{n}}) \left(\hbar\frac{\partial}{\partial\,t} + \frac{\boldsymbol{p}_{\boldsymbol{n}}^2}{2\,m} - \mu\right) \varphi^*\,(t, \boldsymbol{p}_{\boldsymbol{n}}) \qquad (5.98)$$

$$= \sum_{\boldsymbol{n}} S_{0\,\boldsymbol{n}}\!\left(\varphi^*, \overline{\varphi}^*\right)$$

where

$$S_{0\,\boldsymbol{n}}\!\left(\varphi^*, \overline{\varphi}^*\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t\;\, \overline{\varphi}^*(t, \boldsymbol{p}_{\boldsymbol{n}}) \left(\hbar\frac{\partial}{\partial\,t} + \frac{\boldsymbol{p}_{\boldsymbol{n}}^2}{2\,m} - \mu\right) \varphi^*\,(t, \boldsymbol{p}_{\boldsymbol{n}}) \qquad (5.98c)$$

$$\rightarrow \qquad \mathcal{Z}_0(\beta) = \prod_{\boldsymbol{n}} \left(\int [\, d\, \overline{\varphi}^*(t,\, \boldsymbol{p_n})\, d\, \varphi^*(t,\, \boldsymbol{p_n})\,]\, e^{-S_{0\,\boldsymbol{n}}(\varphi^*, \overline{\varphi}^*)/\hbar} \right)$$
$$= \prod_{\boldsymbol{n}} \mathcal{Z}_{0\,\boldsymbol{n}}(\beta)$$

After integrating by part once the time derivative term, $S_{0\,\boldsymbol{n}}$ takes the form of eq(5.25) in §5.2. $\mathcal{Z}_{0\,\boldsymbol{n}}(\beta)$ is therefore given by eq(5.33) so that

$$\mathcal{Z}_0(\beta) = \prod_{\boldsymbol{n}} \left\{ 1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}_{\boldsymbol{n}}^2}{2\,m} - \mu \right) \right] \right\}^{-1} \qquad (5.98d)$$

Since the proper name for $\mathcal{Z}_0$ is the grand partition function, the associated “free energy” has independent variables $(\beta,\, L^d,\, \mu)$ & is called the grand potential $\mathcal{W}_0$ [see e.g., §9.D.4 of L.E.Reichl, “A Modern Course in Statistical Physics”].

$$\mathcal{W}_0(\beta) = -\frac{1}{\beta} \ln \mathcal{Z}_0(\beta)$$
$$= \frac{1}{\beta} \sum_{\boldsymbol{n}} \ln \left\{ 1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}_{\boldsymbol{n}}^2}{2\,m} - \mu \right) \right] \right\} \qquad (5.99)$$
$$= \frac{L^d}{\beta} \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,\hbar)^d} \ln \left\{ 1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right] \right\} \qquad \text{[Eq(5.98b) used]}$$

Note that eq(5.99) differs from Zinn-Justin’s version by a minus sign.

Using [see §2.F.5, Reichl]

$$\mathcal{W}_0 = \Pi\, L^d$$

where Π is the pressure, we have

$$\Pi = \frac{1}{L^d\,\beta} \sum_{\boldsymbol{n}} \ln \left\{ 1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}_{\boldsymbol{n}}^2}{2\,m} - \mu \right) \right] \right\}$$
$$= \frac{1}{\beta} \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,\hbar)^d} \ln \left\{ 1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right] \right\} \qquad \text{[Eq(5.98b) used]} \qquad (5.100)$$

The average number of particles is

$$\langle N \rangle = -\left(\frac{\partial\, \mathcal{W}_0}{\partial\, \mu} \right)_{\beta,\, L} = L^d \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,\hbar)^d} \frac{\exp\left[-\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right]}{1 - \exp\left[-\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right]}$$
$$= L^d \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,\hbar)^d} \frac{1}{\exp\left[\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right] - 1}$$
$$= L^d \int \frac{d\,\boldsymbol{p}}{(2\,\pi\,\hbar)^d}\, n(\boldsymbol{p})$$

where

$$n(\boldsymbol{p}) = \frac{1}{\exp\left[\beta \left(\frac{\boldsymbol{p}^2}{2\,m} - \mu \right) \right] - 1} \qquad (5.102a)$$

gives the average number of particles with momentum $\boldsymbol{p}$.

The average density is therefore

$$\rho = \frac{\langle N \rangle}{L^d} = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{1}{\exp\left[\beta\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)\right] - 1} \tag{5.102}$$

$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} n(\boldsymbol{p})$$

The averge energy density is given by

$$\frac{\langle H \rangle}{L^d} = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} n(\boldsymbol{p}) \frac{\boldsymbol{p}^2}{2m}$$

$$= \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{\frac{\boldsymbol{p}^2}{2m}}{\exp\left[\beta\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)\right] - 1} \tag{5.101}$$

Since $n(\boldsymbol{p})$ denotes number of particles, it must be non-negative. Eq(5.102a) thus gives

$$\exp\left[\beta\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)\right] \geq 1 \qquad \forall\ \boldsymbol{p}$$

$$\to \qquad \beta\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right) \geq 0$$

Since $\beta \geq 0$ & the minimum of $\boldsymbol{p}^2$ is 0, we have

$$\mu \leq 0 \tag{5.101a}$$

From eq(5.102a), we see that if μ is fixed, then $n(\boldsymbol{p})$ decreases as β increases.
Also, for β fixed, $n(\boldsymbol{p})$ increases if μ decreases.

Thus, to keep $\langle N \rangle$ or ρ fixed, μ must decrease as the temperature $T = \frac{1}{k_B \beta}$ lowers.

However, the minimum of μ is 0, which gives a maximum density

$$\rho_{\max} = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \frac{1}{\exp\left(\beta \frac{\boldsymbol{p}^2}{2m}\right) - 1}$$

that the gas could attain at a given β.

Setting $\boldsymbol{x}^2 = \beta \frac{\boldsymbol{p}^2}{2m}$, we have

$$\rho_{\max} = \left(\frac{m}{2\pi^2\hbar^2\beta}\right)^{d/2} \int d\boldsymbol{x}\, \frac{1}{e^{\boldsymbol{x}^2} - 1}$$

$$= \left(\frac{m}{2\pi^2\hbar^2\beta}\right)^{d/2} \int d\boldsymbol{x}\, \frac{e^{-\boldsymbol{x}^2}}{1 - e^{-\boldsymbol{x}^2}}$$

$$= \left(\frac{m}{2\pi^2\hbar^2\beta}\right)^{d/2} \sum_{n=0}^{\infty} \int d\boldsymbol{x}\, e^{-(1+n)\boldsymbol{x}^2}$$

$$= \left(\frac{m}{2\pi^2\hbar^2\beta}\right)^{d/2} \sum_{n=0}^{\infty} \left(\int dx\, e^{-(1+n)x^2}\right)^d$$

$$= \left(\frac{m}{2\pi^2\hbar^2\beta}\right)^{d/2} \sum_{n=0}^{\infty} \left(\frac{\pi}{n+1}\right)^{d/2}$$

$$= \left(\frac{m}{2\pi\hbar^2\beta}\right)^{d/2} \sum_{n=1}^{\infty} \frac{1}{n^{d/2}}$$
$$= \left(\frac{m}{2\pi\hbar^2\beta}\right)^{d/2} \zeta\left(\frac{d}{2}\right)$$
$$= \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{d/2} \zeta\left(\frac{d}{2}\right) \qquad (5.101b)$$

where ζ is Riemann's zeta function.

Consider a system with density ρ. As T lowers below a critical temperature

$$T_0 = \frac{2\pi\hbar^2}{m k_B}\left(\frac{\rho}{\zeta(d/2)}\right)^{2/d}$$

only

$$N_g = \rho_{max}(\beta) L^d$$

particles can stay in the gas state and obeys the distribution governed by $\mathcal{Z}_0(\beta)$. The rest must condense to the ground state of $\boldsymbol{p} = 0$. This phenomenon is called the Bose-Einstein condensation.

However, since $\zeta(1)$ diverges, so does ρ_{max} for $d = 2$. Therefore, no condensation is expected for a 2-D gas.

General One-Body Potential

If we replace the Fourier transform eq(5.96) with the expansion in terms of the eigenfunctions $\{\phi_\alpha\}$ of the 1-body Hamiltonian $\boldsymbol{H}_1$,

$$\varphi(t, \boldsymbol{x}) = \sum_\alpha \phi_\alpha(\boldsymbol{x})\, \varphi^*(t, \alpha)$$
$$\overline{\varphi}(t, \boldsymbol{x}) = \sum_\alpha \overline{\phi}_\alpha(\boldsymbol{x})\, \overline{\varphi}^*(t, \alpha)$$

then everything goes as before with the replacements

$$\sum_{\boldsymbol{n}} \to \sum_\alpha \qquad\qquad \frac{\boldsymbol{p}^2}{2m} \to E_\alpha = \langle \alpha \mid \boldsymbol{H}_1 \mid \alpha \rangle$$

Hence, eq(5.99) becomes

$$\mathcal{W}(\beta) = \frac{1}{\beta} \sum_\alpha \ln\{1 - \exp[-\beta(E_\alpha - \mu)]\}$$
$$= \frac{1}{\beta} \operatorname{tr} \ln\{1 - \exp[-\beta(\boldsymbol{H}_1 - \mu)]\} \qquad (5.103a)$$

Eq(5.102) gives

$$\langle N \rangle = \sum_\alpha \frac{1}{\exp[\beta(E_\alpha - \mu)] - 1} = \sum_\alpha n(E_\alpha)$$
$$= \operatorname{tr} \frac{1}{\exp[\beta(\boldsymbol{H}_1 - \mu)] - 1} \qquad (5.103)$$

Eq(5.101) becomes

$$\langle H \rangle = \sum_\alpha n(E_\alpha)\, E_\alpha$$
$$= \operatorname{tr}\left(\boldsymbol{H}_1 \frac{1}{\exp[\beta(\boldsymbol{H}_1 - \mu)] - 1}\right) \qquad (5.104)$$

Note that Zinn-Justin's version of eqs(5.103-4) differs from ours.

Using the plane waves (or momentum eigenstates) as basis functions, the trace of operator O can be written as

$$\operatorname{tr} O = \int d\boldsymbol{p}\, \langle \boldsymbol{p} \mid O(\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}) \mid \boldsymbol{p} \rangle$$
$$= \int d\boldsymbol{p} \int d\boldsymbol{x} \langle \boldsymbol{p} \mid \boldsymbol{x} \rangle O\left(\boldsymbol{x}, \frac{\hbar}{i}\nabla\right) \langle \boldsymbol{x} \mid \boldsymbol{p} \rangle$$
$$= \int \frac{d\boldsymbol{x}\, d\boldsymbol{p}}{(2\pi\hbar)^d} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar} O\left(\boldsymbol{x}, \frac{\hbar}{i}\nabla\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}$$

where

$$\langle \boldsymbol{x} \mid \boldsymbol{p} \rangle = \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar}}{(2\pi\hbar)^{d/2}}$$

so that

$$\langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle = \int d\boldsymbol{x} \langle \boldsymbol{p} \mid \boldsymbol{x} \rangle \langle \boldsymbol{x} \mid \boldsymbol{p}' \rangle$$
$$= \int \frac{d\boldsymbol{x}}{(2\pi\hbar)^d} e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{x}/\hbar}$$
$$= \delta(\boldsymbol{p}' - \boldsymbol{p})$$

$$\therefore \quad \operatorname{tr} O = \int \frac{d\boldsymbol{x}\, d\boldsymbol{p}}{(2\pi\hbar)^d} O(\boldsymbol{x}, \boldsymbol{p}) \tag{5.104a}$$

Eq(5.103a) can therefore be written as

$$\mathcal{W}(\beta) = \frac{1}{\beta} \int \frac{d\boldsymbol{x}\, d\boldsymbol{p}}{(2\pi\hbar)^d} \ln\left(1 - \exp\left\{-\beta\left[\boldsymbol{H}_1(\boldsymbol{x}, \boldsymbol{p}) - \mu\right]\right\}\right) \tag{5.104b}$$

where $\boldsymbol{H}_1(\boldsymbol{x}, \boldsymbol{p})$ is just the classical Hamiltonian.

A simple example is the spherical harmonic oscillator (or trap)

$$H_1 = \frac{1}{2m}\boldsymbol{p}^2 + \frac{1}{2} m\,\omega^2 \boldsymbol{x}^2$$

Thus,

$$\langle N \rangle = \sum_{n_1 \ldots n_d} 1 \Big/ \left(\exp\left\{\beta\left[\left(n_1 + \ldots + n_d + \frac{d}{2}\right)\hbar\omega - \mu\right]\right\} - 1\right)$$

$$= \sum_{n_1 \ldots n_d} \left(\exp\left\{-\beta\left[\left(n_1 + \ldots + n_d + \frac{d}{2}\right)\hbar\omega - \mu\right]\right\}\right) \Big/ \left(1 - \exp\left\{-\beta\left[\left(n_1 + \ldots + n_d + \frac{d}{2}\right)\hbar\omega - \mu\right]\right\}\right)$$

$$= \sum_{m=0}^{\infty} \sum_{n_1 \ldots n_d} \exp\left\{-\beta(m+1)\left[\left(n_1 + \ldots + n_d + \frac{d}{2}\right)\hbar\omega - \mu\right]\right\}$$

In the classical approximation, we drop the term $d/2$ & approximate the n_j sums by integrals. If furthermore, $\mu = 0$, we have

$$\langle N \rangle \approx \sum_{m=0}^{\infty} \int d n_1 \ldots d n_d \exp\left\{-\beta\hbar\omega(m+1)(n_1 + \ldots + n_d)\right\}$$
$$= \sum_{m=1}^{\infty} \left(\int_0^{\infty} d n \exp(-\hbar\omega m n)\right)^d$$
$$= \sum_{m=1}^{\infty} \left(\frac{1}{\beta\hbar\omega m}\right)^d$$

$$= \frac{\zeta(d)}{(\beta\hbar\omega)^d} \tag{5.104b}$$

which should be valid for high $T >> \frac{\hbar\omega}{k_B}$.

For a system with average particle number $\langle N \rangle$, the critical temperature is

$$T_0 = \frac{\hbar\omega}{k_B}\left(\frac{\langle N \rangle}{\zeta(d)}\right)^{1/d}$$

so that Bose condensation is possible for any $d > 1$.

If we wish to include the zero-point contribution, we can set $\mu = \frac{d}{2}\hbar\omega$ to regain the result eq(5.104a).

5.5.4. Pair-Potentials: An Example

The δ(x)-Function Potential

For particles interacting with a delta function potential,

$$V_2(\boldsymbol{x}, \boldsymbol{y}) = g\,\delta(\boldsymbol{x} - \boldsymbol{y})$$

eq(5.90w) becomes

$$\hat{\boldsymbol{V}}_2 \langle \overline{\varphi} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \frac{1}{2} g \int d\boldsymbol{x}\, \overline{\varphi}^2(\boldsymbol{x})\, \varphi'^2(\boldsymbol{x})$$

Setting

$$V_1(\boldsymbol{x}) = 0$$

the action eq(5.95) becomes

$$S(\varphi, \overline{\varphi}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x} \left\{ \overline{\varphi}\left(\hbar\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 - \mu\right)\varphi + \frac{1}{2} g\, \overline{\varphi}^2 \varphi^2 \right\} \tag{5.105}$$

where $\varphi = \varphi(t, \boldsymbol{x})$ in the integrand is understood.

Let $[A]$ denotes the dimension of A. Then

$$[S] = [\hbar] \qquad\qquad [\beta] = [H]^{-1}$$

$$[d\boldsymbol{x}\,\overline{\varphi}\,\varphi] = [1] \quad \to \quad [\overline{\varphi}\,\varphi] = [L]^{-d}$$

$$[g\,\overline{\varphi}^2\varphi^2] = [\overline{\varphi}\,H\,\varphi]$$

$$\to \quad [g] = \left[\frac{H}{\overline{\varphi}\,\varphi}\right] = [H L^d] = \left[\frac{\hbar^2}{m a^2} a^d\right] = \left[\frac{\hbar^2}{m} a^{d-2}\right]$$

where a is the scattering length.

See Zinn-Justin's text for discussion.

Perturbation Theory

As an exercise, we calculate the correction of order g to the free energy.

The Euler-Lagrange eq. for the action eq(5.105) is obtained as follows [see §"Generating Functional" of §5.2].

Remembering that the variable corresponding to z is $\overline{\varphi}$, we begin by integrating by part to simplify the functional derivative calculations:

$$S(\varphi, \overline{\varphi}) = \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x} \left\{ \varphi\left(-\hbar\frac{\partial}{\partial t} - \mu\right)\overline{\varphi} + \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}\varphi\cdot\nabla_{\boldsymbol{x}}\overline{\varphi} + \frac{1}{2} g\,\overline{\varphi}^2\varphi^2 \right\} \tag{5.105a}$$

Let

$$S = \int d\,t\,L = \int d\,t \int d\,\boldsymbol{x}\,\mathcal{L}$$

$$\rightarrow \qquad \mathcal{L} = \varphi\left(-\hbar\frac{\partial}{\partial t} - \mu\right)\overline{\varphi} + \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}\varphi\cdot\nabla_{\boldsymbol{x}}\overline{\varphi} + \frac{1}{2}g\,\overline{\varphi}^2\varphi^2$$

$$\frac{\partial\mathcal{L}}{\partial\overline{\varphi}} = -\mu\,\varphi + g\,\varphi^2\,\overline{\varphi} \qquad \frac{\partial\mathcal{L}}{\partial\dot{\overline{\varphi}}} = -\hbar\,\varphi \qquad \frac{\partial\mathcal{L}}{\partial\nabla_{\boldsymbol{x}}\overline{\varphi}} = \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}\varphi$$

$$\rightarrow \qquad \left(-\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 + \mu - g\,\overline{\varphi}\,\varphi\right)\varphi = 0 \qquad (5.105b)$$

The corresponding green’s function is given by [see eq(5.31t) of §5.2]

$$\dot{\Delta}(t,\boldsymbol{x}) - \left(\frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 + \mu\right)\Delta(t,\boldsymbol{x}) = \delta(t) \qquad (5.105c)$$

Taking the Fourier transform of eq(5.105b), we have

$$\left(\hbar\frac{\partial}{\partial t} - \mu\right)\varphi^* + \frac{\boldsymbol{p}^2}{2m}\varphi^* + g\,\overline{\varphi}^*\,\varphi^{*\,2} = 0 \qquad \varphi^* = \varphi^*(t,\boldsymbol{p}) = \varphi_c^*$$

or $$\dot{\varphi}_c^* + \omega(\boldsymbol{p})\,\varphi_c^* = -\frac{g}{\hbar}\,\overline{\varphi}^*\,\varphi^{*\,2} \qquad (5.105d)$$

where $\omega(\boldsymbol{p}) = \frac{1}{\hbar}\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)$.

The corresponding green’s function is given by [see eq(5.31t) of §5.2]

$$\dot{\Delta}(t,\boldsymbol{p}) + \omega(\boldsymbol{p})\,\Delta(t,\boldsymbol{p}) = \delta(t) \qquad (5.105e)$$

$$\rightarrow \qquad \Delta(t,\boldsymbol{p}) = \frac{e^{-\omega(\boldsymbol{p})\,t}}{2}\left[\epsilon(t) + \frac{1 + e^{-\beta\hbar\,\omega(\boldsymbol{p})}}{1 - e^{-\beta\hbar\,\omega(\boldsymbol{p})}}\right] \qquad \text{[See eq(5.30)]} \qquad (5.105f)$$

Using

$$\theta(t) = \frac{1}{2}[\epsilon(t) + 1] = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

eq(5.105f) becomes

$$\begin{aligned}\Delta(t,\boldsymbol{p}) &= e^{-\omega(\boldsymbol{p})\,t}\left[\theta(t) + \frac{1}{2}\left(-1 + \frac{1 + e^{-\beta\hbar\,\omega(\boldsymbol{p})}}{1 - e^{-\beta\hbar\,\omega(\boldsymbol{p})}}\right)\right] \\ &= e^{-\omega(\boldsymbol{p})\,t}\left[\theta(t) + \frac{e^{-\beta\hbar\,\omega(\boldsymbol{p})}}{1 - e^{-\beta\hbar\,\omega(\boldsymbol{p})}}\right] \\ &= e^{-\omega(\boldsymbol{p})\,t}\left[\theta(t) + \frac{1}{e^{\beta\hbar\,\omega(\boldsymbol{p})} - 1}\right] \qquad (5.105g)\end{aligned}$$

The semi-classical approximation [see eq(5.32)] of setting

$$\varphi^*(t,\boldsymbol{p}) = \varphi_c^*(t,\boldsymbol{p}) + r^*(t,\boldsymbol{p}) \qquad \text{or} \qquad \varphi(t,\boldsymbol{x}) = \varphi_c(t,\boldsymbol{x}) + r(t,\boldsymbol{x})$$

doesn’t work since

$$S_L(\varphi_c^*, r^*) \neq 0 \qquad \& \qquad S_L(\varphi_c, r) \neq 0$$

so that $S(\varphi^*, \overline{\varphi}^*)$ & $S(\varphi, \overline{\varphi})$ depends explicitly on r^* & r, respectively.

Nevertheless, one can follow §5.2 & introduce a $b\,\overline{\varphi} + \overline{b}\,\varphi$ term to show that

$$\langle\,\overline{\varphi}^*(t,\boldsymbol{p})\,\varphi^*(0,\boldsymbol{p}')\,\rangle_0 = \delta(\boldsymbol{p} - \boldsymbol{p}')\,\Delta(t,\boldsymbol{p}) \qquad (5.105h)$$

Perturbation can still be done by expanding

$$e^{-S/\hbar} = e^{-S_0/\hbar}\sum_n \frac{1}{n!}\mathcal{V}_2^n$$

where

$$\mathcal{V}_2 = -\frac{1}{\hbar}\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\mathbf{x} \, \frac{1}{2} g \overline{\varphi}^2 \varphi^2$$

Thus,

$$\mathcal{W}(\beta) = \mathcal{W}_0(\beta) - \frac{g}{2\hbar}\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\mathbf{x} \langle \overline{\varphi}^2 \varphi^2 \rangle_0 + O(g^2)$$

$$\int d\mathbf{x} \langle \overline{\varphi}^2 \varphi^2 \rangle_0 = \int d\mathbf{x} \langle \overline{\varphi}(t, \mathbf{x}) \overline{\varphi}(t, \mathbf{x}) \varphi(t, \mathbf{x}) \varphi(t, \mathbf{x}) \rangle_0$$

$$= \int d\mathbf{x} \int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2 \, d\mathbf{p}_3 \, d\mathbf{p}_4}{(2\pi\hbar)^{2d}} e^{i(-\mathbf{p}_1 - \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4)\cdot \mathbf{x}/\hbar}$$
$$\times \langle \overline{\varphi}(t, \mathbf{p}_1) \overline{\varphi}(t, \mathbf{p}_2) \varphi(t, \mathbf{p}_3) \varphi(t, \mathbf{p}_4) \rangle_0$$

$$= \int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2 \, d\mathbf{p}_3 \, d\mathbf{p}_4}{(2\pi\hbar)^{d}} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)$$
$$\times \langle \overline{\varphi}(t, \mathbf{p}_1) \overline{\varphi}(t, \mathbf{p}_2) \varphi(t, \mathbf{p}_3) \varphi(t, \mathbf{p}_4) \rangle_0$$

To apply Wick's theorem, let $\varphi_j = \varphi(t, \mathbf{p}_j)$ so that

$$\langle \overline{\varphi}_1 \overline{\varphi}_2 \varphi_3 \varphi_4 \rangle_0 = \langle \overline{\varphi}_1 \varphi_3 \rangle_0 \langle \overline{\varphi}_2 \varphi_4 \rangle_0 + \langle \overline{\varphi}_1 \varphi_4 \rangle_0 \langle \overline{\varphi}_2 \varphi_3 \rangle_0$$
$$= \delta(\mathbf{p}_1 - \mathbf{p}_3)\,\delta(\mathbf{p}_2 - \mathbf{p}_4)\,\Delta(0, \mathbf{p}_1)\,\Delta(0, \mathbf{p}_2)$$
$$+\delta(\mathbf{p}_1 - \mathbf{p}_4)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\Delta(0, \mathbf{p}_1)\,\Delta(0, \mathbf{p}_2)$$

$$\rightarrow \quad \int d\mathbf{x} \langle \overline{\varphi}^2 \varphi^2 \rangle_0 = 2\int \frac{d\mathbf{p}_1 \, d\mathbf{p}_2}{(2\pi\hbar)^{d}} \Delta(0, \mathbf{p}_1)\,\Delta(0, \mathbf{p}_2)$$
$$= 2\,(2\pi\hbar)^d \left(\int \frac{d\mathbf{p}}{(2\pi\hbar)^d} \Delta(0, \mathbf{p}) \right)^2$$

$$\rightarrow \quad \mathcal{W}(\beta) = \mathcal{W}_0(\beta) - g\,(2\pi\hbar)^d \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \left(\int \frac{d\mathbf{p}}{(2\pi\hbar)^d} \Delta(0, \mathbf{p}) \right)^2 + O(g^2)$$

$$= \mathcal{W}_0(\beta) - g\,\beta\,(2\pi\hbar)^d \left(\int \frac{d\mathbf{p}}{(2\pi\hbar)^d} \Delta(0, \mathbf{p}) \right)^2 + O(g^2) \qquad (5.105i)$$

Using eq(5.105g) to evaluate $\Delta(0, \mathbf{p})$, we encounter the $\epsilon(0)$ problem again. Choosing $\theta(0) = 0$, we have

$$\Delta(0, \mathbf{p}) = \frac{1}{e^{\beta\hbar\omega(\mathbf{p})} - 1}$$

High Temperature

Periodic B.C. eq(5.95a) means that φ can be written as a Fourier series in time:

$$\varphi(t, \mathbf{x}) = \sum_{\nu \in \mathbb{Z}} e^{2\pi i \nu t/\beta} \varphi_\nu(\mathbf{x}) \qquad \overline{\varphi}(t, \mathbf{x}) = \sum_{\nu \in \mathbb{Z}} e^{-2\pi i \nu t/\beta} \overline{\varphi}_\nu(\mathbf{x}) \quad (5.106a)$$

As $\beta \to 0$, $e^{2\pi i \nu t/\beta}$ fluctuates wildly except for the t-independent $\nu = 0$ term.

A more practical indication of this high-temperature region is

$$\lambda_{th} = \hbar\sqrt{\frac{\beta}{m}} = \sqrt{\frac{\hbar^2}{m k_B T}} << \xi \qquad (5.106b)$$

where λ_{th} & ξ are the thermal & coherence lengths, respectively.

Given eq(5.106b) is satisfied, the action eq(5.105) can be approximated by

$$S(\varphi, \overline{\varphi}) = \int d\,\boldsymbol{x} \left\{ \overline{\varphi}_0 \left(-\frac{\hbar^2}{2\,m} \nabla_{\boldsymbol{x}}^2 - \mu \right) \varphi_0 + \frac{1}{2}\, g\, \overline{\varphi}_0^{\,2}\, \varphi_0^2 \right\} \tag{5.106}$$