5.6. The Fermi Gas. Functional Integrals

This section is the straight-forward many body generalization of §5.3.
All derivations follow closely those of the Boson gas given in §5.5. One need only take care to observe the anti-symmetric rule.

Consider the Grassmann algebra with generators $\varphi(\boldsymbol{x})$ such that

$$\varphi(\boldsymbol{x})\,\varphi(\boldsymbol{x}') + \varphi(\boldsymbol{x}')\,\varphi(\boldsymbol{x}) = 0$$

Starting with the same $\boldsymbol{H}$ given by eq(5.82), we define the functional [see eq(5.86)]

$$\boldsymbol{\Psi}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \left(\prod_{i=1}^{n} d^d x_i\, \varphi(\boldsymbol{x}_i) \right) \psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \qquad (5.107)$$

where $\psi_n(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is anti-symmetric in its arguments, i.e., ψ_n is a linear combination of $n!$ terms of equal weight but alternating signs.

Following the derivations of eq(5.90), we get

$$\langle \overline{\varphi} \mid \boldsymbol{H} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \left\{ \int d\boldsymbol{x}\; \overline{\varphi}(\boldsymbol{x}) \left[-\frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) \right] \varphi'(\boldsymbol{x}) \right. \qquad (5.108)$$

$$\left. + \frac{1}{2} \int d\boldsymbol{x}\, d\boldsymbol{y}\; \overline{\varphi}(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi'(\boldsymbol{y})\, \varphi'(\boldsymbol{x}) \right\}$$

where, generalizing eq(5.45c) of §5.3,

$$\langle \overline{\varphi} \mid \overline{\varphi}' \rangle = -\int d^d x\, \varphi'(\boldsymbol{x})\, \overline{\varphi}(\boldsymbol{x})$$

Following the derivations of eq(5.95), with the generator ordering given by eq(5.66), we get the partition function,

$$\mathcal{Z}(\beta) = \int [\, d\,\varphi(t, \boldsymbol{x})\, d\,\overline{\varphi}(t, \boldsymbol{x})\,]\, e^{-S(\varphi, \overline{\varphi})} \qquad (5.109)$$

$$S(\varphi, \overline{\varphi}) = -\int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x}\; \overline{\varphi}(t, \boldsymbol{x}) \left(\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi(t, \boldsymbol{x}) \qquad (5.110)$$

$$+ \frac{1}{2} \int_{-\hbar\beta/2}^{\hbar\beta/2} dt \int d\boldsymbol{x}\, d\boldsymbol{y}\; \overline{\varphi}(t, \boldsymbol{x})\, \overline{\varphi}(t, \boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi(t, \boldsymbol{y})\, \varphi(t, \boldsymbol{x})$$

where the overall "–" sign in the 1st term in eq(5.110) is due to the fact that $\overline{\varphi}$ was originally on the right of φ in the functional derivative formulas.

Finally, the B.C. are anti-periodic:

$$\varphi(\hbar\beta/2, \boldsymbol{x}) = -\varphi(-\hbar\beta/2, \boldsymbol{x}) \qquad \overline{\varphi}(\hbar\beta/2, \boldsymbol{x}) = -\overline{\varphi}(-\hbar\beta/2, \boldsymbol{x}) \quad (5.110a)$$

5.6.1. Simple Examples

The Free Fermi Gas

Since for each degrees of freedom of a free Fermion, we have [see eq(5.73)]

$$\mathcal{Z}_0(\beta) = 1 + e^{-\beta\hbar\Omega} \qquad \hbar\Omega(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m} - \mu$$

therefore, for a free Fermi gas, we have [see eq(5.98d)]

$$\mathcal{Z}_0(\beta) = \prod_{\boldsymbol{n}} \left(1 + e^{-\beta\hbar\Omega(\boldsymbol{p}_n)} \right)$$

$$\to \quad \mathcal{W}(\beta) = -\frac{1}{\beta} \ln \mathcal{Z}_0(\beta)$$
$$= -\frac{1}{\beta} \sum_{\boldsymbol{n}} \ln\left(1 + e^{-\beta \hbar \Omega(\boldsymbol{p}_n)}\right)$$
$$= -\frac{L^d}{\beta} \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} \ln\left(1 + e^{-\beta \hbar \Omega(\boldsymbol{p}_n)}\right) \qquad (5.111)$$

Eq(5.102a) becomes

$$n(\boldsymbol{p}) = \frac{1}{e^{\beta \hbar \Omega(\boldsymbol{p})} + 1} \qquad (5.112a)$$
$$= \frac{1}{\exp\left[\beta\left(\frac{\boldsymbol{p}^2}{2m} - \mu\right)\right] + 1}$$

so that the average density is

$$\rho = \frac{\langle N \rangle}{L^d} = \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} n(\boldsymbol{p})$$
$$= \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} \frac{1}{e^{\beta\hbar\Omega(\boldsymbol{p})} + 1} \qquad (5.112)$$

& the average energy density is

$$\frac{\langle H \rangle}{L^d} = \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} n(\boldsymbol{p}) \frac{\boldsymbol{p}^2}{2m}$$
$$= \int \frac{d\,\boldsymbol{p}}{(2\pi\hbar)^d} \frac{1}{e^{\beta\hbar\Omega(\boldsymbol{p})} + 1} \left(\frac{\boldsymbol{p}^2}{2m}\right) \qquad (5.113)$$

The δ(x)-Function Potential

Owing to the exclusion principle, contact interactions examplified by the delta function potential vanish unless the fermions possess internal degrees of freedom.

A simple example is the N-component field ψ^α in an action with $U(N)$ symmetry:

$$S(\varphi, \overline{\varphi}) = -\int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,\boldsymbol{x} \left\{ \overline{\varphi}^\alpha(t, \boldsymbol{x}) \left(\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi^\alpha(t, \boldsymbol{x}) \right. \qquad (5.114)$$
$$\left. + \frac{1}{2} g \left[\sum_\alpha \overline{\varphi}^\alpha(t, \boldsymbol{x})\, \varphi^\alpha(t, \boldsymbol{x}) \right]^2 \right\}$$

See Zinn-Justin's text for comments.

5.6.2. Non-Relativistic Fermion Gas at Low Temperature, in One Dimension

The occupation number eq(5.112a) becomes a step function at $T = 0$ so that all states below μ are occupied while those above are empty. The chemical potential μ is often called the Fermi energy E_F. The Fermi momentum $\boldsymbol{p}_F$ and wave vector $\boldsymbol{k}_F$ are given by

$$\mu = E_F = \frac{\boldsymbol{p}_F^2}{2m} = \frac{\hbar^2 \boldsymbol{k}_F^2}{2m} \qquad (5.115)$$

At low T (or $\beta >> 1$), excitations occur mainly within a few $\frac{1}{\beta} = k_B T$ near E_F.

Similarly, in the presence of weak interactions, only states near E_F, with momenta $|p| \approx \hbar k_F$, are significantly disturbed.

In 1-D, this means we can set

$$k = \pm k_F + q \qquad |\, q \,| << k_F \qquad (5.115a)$$

The green function $\Delta(t, k)$ satisfies the same equation as the bosons [see eq(5.105e)]

$$\dot{\Delta}(t, k) + \Omega(k)\,\Delta(t, k) = \delta(t) \qquad \hbar\,\Omega(k) = \frac{\hbar^2 k^2}{2m} - \mu \qquad (5.115b)$$

but obeys anti-periodic B.C. Taking the Fourier transform in t, we have

$$\Delta(t, k) = \sum_{n=-\infty}^{\infty} \exp(i\,\omega_n t)\,\Delta(\omega_n, k) \qquad \omega_n = \frac{2\pi\,(n+1/2)}{\hbar\beta}$$

$$\rightarrow \qquad \Delta(\beta\hbar/2, k) = \sum_{n=-\infty}^{\infty} \exp\left[i\left(n+\frac{1}{2}\right)\pi\right]\Delta(\omega_n, k) = \sum_{n=-\infty}^{\infty} i\,(-)^n\,\Delta(\omega_n, k)$$

$$\Delta(-\beta\hbar/2, k) = \sum_{n=-\infty}^{\infty} \exp\left[-i\left(n+\frac{1}{2}\right)\pi\right]\Delta(\omega_n, k) = -\sum_{n=-\infty}^{\infty} i\,(-)^n\,\Delta(\omega_n, k)$$

$$= -\Delta(\beta\hbar/2, k) \qquad \text{(as required by the anti-periodic B.C.)}$$

Using

$$\delta(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \qquad \text{for} \qquad x \in (-1, 1)$$

we have

$$\delta(t) = \sum_{n=-\infty}^{\infty} e^{2\pi i \frac{n}{\beta\hbar} t} \qquad \text{for} \qquad t \in (-\beta\hbar/2, \beta\hbar/2)$$

$$= e^{-i\pi t/\hbar\beta} \sum_{n=-\infty}^{\infty} e^{i\,\omega_n t}$$

Since $\delta(t) \neq 0$ only when $t = 0$, we can set

$$e^{-i\pi t/\hbar\beta} \rightarrow e^{-i\pi t/\hbar\beta}\Big|_{t=0} = 1$$

$$\therefore \qquad \delta(t) = \sum_{n=-\infty}^{\infty} e^{i\,\omega_n t}$$

Eq(5.115b) becomes

$$\sum_n \left\{ [\,i\,\omega_n + \Omega(k)\,]\,\Delta(\omega_n, k) - 1 \right\} \exp(i\,\omega_n t) = 0$$

$$\rightarrow \qquad \Delta(\omega_n, k) = \frac{1}{i\,\omega_n + \Omega(k)}$$

$$= \frac{\hbar}{i\,\hbar\,\omega_n + \frac{\hbar^2 k^2}{2m} - \mu} \qquad (5.115c)$$

$$= \frac{\hbar}{i\,\hbar\,\omega_n + \frac{\hbar^2}{2m}\left(k^2 - k_F^2\right)}$$

The maximum of $|\,\Delta(\omega_n, k)\,|$ is obviously at $k = k_F$.

Using eq(5.115a), we have, for small q,

$$k^2 - k_F^2 \approx \begin{cases} 2k_F q & \text{for } k = k_F + q \\ -2k_F q & \text{for } k = k_F - q \end{cases}$$

Near $k = k_F$, we have

$$\Delta(\omega_n, k) \approx \begin{cases} \dfrac{1}{i\,\omega_n + \dfrac{\hbar\, k_F\, q}{m}} & \text{for } k = k_F + q \\ \dfrac{1}{i\,\omega_n - \dfrac{\hbar\, k_F\, q}{m}} & \text{for } k = k_F - q \end{cases}$$

The expression

$$\Delta\,(\omega_n, k) \approx \frac{1}{i\,\omega_n + \dfrac{\hbar\, k_F\, q}{m}} + \frac{1}{i\,\omega_n - \dfrac{\hbar\, k_F\, q}{m}} \qquad (5.115d)$$

is thus a fairly good approximation except that at $q = 0$, it gives twice the actual value.

Eq(5.115d) has the same form as that for a massless relativistic fermion with the 1st & 2nd term representing motion toward the left & right, respectively. $\dfrac{\hbar\, k_F}{m}$ thus plays the role of the speed of light.

To make it more explicit, let

$$\varphi(t, x) = e^{-i\, k_F\, x}\, \eta_1(t, x) + e^{i\, k_F\, x}\, \eta_2(t, x) \qquad (5.115e)$$
$$\overline{\varphi}\,(t, x) = -i\, e^{-i\, k_F\, x}\, \tilde{\eta}_1(t, x) + i\, e^{i\, k_F\, x}\, \tilde{\eta}_2(t, x)$$

i.e., $\quad \tilde{\eta}_1 = i\, \overline{\eta}_2 \qquad\qquad \tilde{\eta}_2 = -i\, \overline{\eta}_1$

Caution: In Zinn-Justin's text, $\tilde{\eta}$ is denoted rather confusingly as $\overline{\eta}$.

$\to$
$$\dot{\varphi} = e^{-i\, k_F\, x}\, \dot{\eta}_1 + e^{i\, k_F\, x}\, \dot{\eta}_2$$
$$\partial_x \varphi = i\, k_F\left(-e^{-i\, k_F\, x}\, \eta_1 + e^{i\, k_F\, x}\, \eta_2\right) + e^{-i\, k_F\, x}\, \partial_x \eta_1 + e^{i\, k_F\, x}\, \partial_x \eta_2$$
$$\partial_x^2 \varphi = -k_F^2\left(e^{-i\, k_F\, x}\, \eta_1 + e^{i\, k_F\, x}\, \eta_2\right) + 2\, i\, k_F\left(-e^{-i\, k_F\, x}\, \partial_x \eta_1 + e^{i\, k_F\, x}\, \partial_x \eta_2\right)$$
$$\quad + e^{-i\, k_F\, x}\, \partial_x^2 \eta_1 + e^{i\, k_F\, x}\, \partial_x^2 \eta_2$$

Putting these into the free action

$$S_0\left(\varphi, \overline{\varphi}\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,x\; \overline{\varphi}(t, x)\left(-\hbar\, \frac{\partial}{\partial\, t} + \frac{\hbar^2}{2\, m}\, \frac{\partial^2}{\partial\, x^2} + \mu\right) \varphi(t, x)$$

& dropping all terms containing a high energy demanding $e^{\pm i\, k_F\, x/\hbar}$ factor, we have

$$S_0\left(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}\right) = \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,x \left\{ i\,\hbar\left(\tilde{\eta}_1\, \dot{\eta}_2 - \tilde{\eta}_2\, \dot{\eta}_1\right) + \frac{\hbar^2\, k_F}{m}\left(\tilde{\eta}_1\, \partial_x \eta_2 + \tilde{\eta}_2\, \partial_x \eta_1\right)\right\}$$

where we've also used $\mu = \dfrac{\hbar^2\, k_F^2}{2\, m}$ & dropped terms involving $\partial_x^2\, \eta_j$.

Using the Pauli matrices σ_j, we have

$$\tilde{\boldsymbol{\eta}}^T\, \sigma_2\, \dot{\boldsymbol{\eta}} = \left(\tilde{\eta}_1, \tilde{\eta}_2\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{pmatrix} = i\left(-\tilde{\eta}_1\, \dot{\eta}_2 + \tilde{\eta}_2\, \dot{\eta}_1\right)$$

$$\tilde{\boldsymbol{\eta}}^T\, \sigma_1\, \partial_x \boldsymbol{\eta} = \left(\tilde{\eta}_1, \tilde{\eta}_2\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \eta_1 \\ \partial_x \eta_2 \end{pmatrix} = \tilde{\eta}_1\, \partial_x \eta_2 + \tilde{\eta}_2\, \partial_x \eta_1$$

so that

$$S_0(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) \approx \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,x \left(-\hbar\, \tilde{\boldsymbol{\eta}}^T\, \sigma_2\, \dot{\boldsymbol{\eta}} + \frac{\hbar^2\, k_F}{m}\, \tilde{\boldsymbol{\eta}}^T\, \sigma_1\, \partial_x \boldsymbol{\eta}\right) \qquad (5.115f)$$

Note that

$$\tilde{\boldsymbol{\eta}}^T \sigma_2 = (\tilde{\eta}_1, \tilde{\eta}_2)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\,(\tilde{\eta}_2, -\tilde{\eta}_1) = (\overline{\eta}_1, \overline{\eta}_2) = \boldsymbol{\eta}^+$$

Although the delta function potential vanishes for an 1-component field φ, it is still useful to express it in terms of the $\boldsymbol{\eta}$ fields under the foregoing approximations:

$$\begin{aligned}(\overline{\varphi}\,\varphi)^2 &= \overline{\varphi}\,\varphi\,\overline{\varphi}\,\varphi \\ &= \left(-i\,e^{-i k_F x}\,\tilde{\eta}_1 + i\,e^{i k_F x}\,\tilde{\eta}_2\right)\left(e^{-i k_F x}\,\eta_1 + e^{i k_F x}\,\eta_2\right) \\ &\quad\times\left(-i\,e^{-i k_F x}\,\tilde{\eta}_1 + i\,e^{i k_F x}\,\tilde{\eta}_2\right)\left(e^{-i k_F x}\,\eta_1 + e^{i k_F x}\,\eta_2\right) \\ &= \left(-i\,e^{-2 i k_F x}\,\tilde{\eta}_1\,\eta_1 - i\,\tilde{\eta}_1\,\eta_2 + i\,\tilde{\eta}_2\,\eta_1 + i\,e^{2 i k_F x}\,\tilde{\eta}_2\,\eta_2\right) \\ &\quad\times\left(-i\,e^{-2 i k_F x}\,\tilde{\eta}_1\,\eta_1 - i\,\tilde{\eta}_1\,\eta_2 + i\,\tilde{\eta}_2\,\eta_1 + i\,e^{2 i k_F x}\,\tilde{\eta}_2\,\eta_2\right) \\ &\approx \tilde{\eta}_1\,\eta_1\,\tilde{\eta}_2\,\eta_2 + \left(-i\,\tilde{\eta}_1\,\eta_2 + i\,\tilde{\eta}_2\,\eta_1\right)^2 + \tilde{\eta}_2\,\eta_2\,\tilde{\eta}_1\,\eta_1 \\ &= \tilde{\eta}_1\,\eta_1\,\tilde{\eta}_2\,\eta_2 + \left(\tilde{\boldsymbol{\eta}}^T\,\sigma_2\,\boldsymbol{\eta}\right)^2 + \tilde{\eta}_2\,\eta_2\,\tilde{\eta}_1\,\eta_1\end{aligned}$$

Using

$$\tilde{\boldsymbol{\eta}}^T \sigma_3\,\boldsymbol{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \tilde{\eta}_1\,\eta_1 - \tilde{\eta}_2\,\eta_2$$

$$\tilde{\boldsymbol{\eta}}^T\,\boldsymbol{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \tilde{\eta}_1\,\eta_1 + \tilde{\eta}_2\,\eta_2$$

we have

$$(\overline{\varphi}\,\varphi)^2 \approx \frac{1}{2}\left[\left(\tilde{\boldsymbol{\eta}}^T\,\boldsymbol{\eta}\right)^2 - \left(\tilde{\boldsymbol{\eta}}^T\,\sigma_3\,\boldsymbol{\eta}\right)^2\right] + \left(\tilde{\boldsymbol{\eta}}^T\,\sigma_2\,\boldsymbol{\eta}\right)^2$$

Eq(5.114) thus becomes

$$\begin{aligned}S(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) \approx \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,x \Big\{ &-\hbar\,\tilde{\boldsymbol{\eta}}^T\,\sigma_2\,\dot{\boldsymbol{\eta}} + \frac{\hbar^2 k_F}{m}\,\tilde{\boldsymbol{\eta}}^T\,\sigma_1\,\partial_x\,\boldsymbol{\eta} \\ &+ \frac{1}{4}\,g\left[\left(\tilde{\boldsymbol{\eta}}^T\,\boldsymbol{\eta}\right)^2 - \left(\tilde{\boldsymbol{\eta}}^T\,\sigma_3\,\boldsymbol{\eta}\right)^2\right] + \frac{1}{2}\,g\left(\tilde{\boldsymbol{\eta}}^T\,\sigma_2\,\boldsymbol{\eta}\right)^2 \Big\}\end{aligned} \qquad (5.115g)$$

For an N-component fermion field $\boldsymbol{\varphi} = \{\varphi^\alpha\}$, we apply eq(5.115e) to each component φ^α to define the 2-component fields $\boldsymbol{\eta}^\alpha = (\eta_1^\alpha, \eta_2^\alpha)$ & $\tilde{\boldsymbol{\eta}}^\alpha = (\tilde{\eta}_1^\alpha, \tilde{\eta}_2^\alpha)$.

Eq(5.115g) is readily generalized to

$$\begin{aligned}S(\boldsymbol{\eta}, \tilde{\boldsymbol{\eta}}) \approx \int_{-\hbar\beta/2}^{\hbar\beta/2} d\,t \int d\,x \sum_\alpha \Big\{ &-\hbar\,\tilde{\boldsymbol{\eta}}^{\alpha T}\,\sigma_2\,\dot{\boldsymbol{\eta}}^\alpha + \frac{\hbar^2 k_F}{m}\,\tilde{\boldsymbol{\eta}}^{\alpha T}\,\sigma_1\,\partial_x\,\boldsymbol{\eta}^\alpha \\ &+ \frac{1}{4}\,g\left[\left(\tilde{\boldsymbol{\eta}}^{\alpha T}\,\boldsymbol{\eta}^\alpha\right)^2 - \left(\tilde{\boldsymbol{\eta}}^{\alpha T}\,\sigma_3\,\boldsymbol{\eta}^\alpha\right)^2\right] + \frac{1}{2}\,g\left(\tilde{\boldsymbol{\eta}}^{\alpha T}\,\sigma_2\,\boldsymbol{\eta}^\alpha\right)^2 \Big\}\end{aligned} \qquad (5.115h)$$

The two-component fermion fields $\{\boldsymbol{\eta}^\alpha, \tilde{\boldsymbol{\eta}}^\alpha\}$ represent N massless Dirac fermions. The action has a chiral symmetry

$$\boldsymbol{\eta}^\alpha \mapsto e^{i\,\sigma_3\,\theta}\,\boldsymbol{\eta}^\alpha \qquad\qquad \tilde{\boldsymbol{\eta}}^{\alpha T} \mapsto \tilde{\boldsymbol{\eta}}^{\alpha T}\,e^{-i\,\sigma_3\,\theta}$$

See Zinn-Justin's text for further comments.