Quantum Evolution and Scattering Matrix

Evolution in quantum mechanics is described by the evolution operator U. Conservation of probabilities requires U to be unitary. Assuming evolution of an isolated system is Markovian (with no memory effects), U satisfies the group properties

$$U(t, t) = 1$$
$$U(t_3, t_2)\, U(t_2, t_1) = U(t_3, t_1) \tag{6.1}$$

The Schroginger eq. requires U to satisfy

$$i\hbar \frac{\partial U(t, t')}{\partial t} = H(t)\, U(t, t') \tag{6.2}$$

Integrating for infinitesimal time increment ε,

$$U(t+\varepsilon, t) = U(t, t) + \varepsilon \left.\frac{\partial U(t', t)}{\partial t'}\right|_{t'=t} + O(\varepsilon^2)$$
$$= 1 - \frac{i}{\hbar}\varepsilon H(t) + O(\varepsilon^2)$$

$\frac{\partial}{\partial t}$ can be considered as a Lie derivative of operators on the Hilbert space, thus giving rise to the Heisenberg picture.

The S-Matrix

The S-matrix is defined as the limit of U in the interaction picture

$$S = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{i H_0 t''/\hbar}\, U(t'', t')\, e^{-i H_0 t'/\hbar} \tag{6.3}$$

6.1. Time Evolution and Scattering Matrix in Quantum Mechanics

For a particle in a potential, the free & interacting hamiltonians are

$$H_0 = \frac{\boldsymbol{p}^2}{2m} \qquad\qquad H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}, t) \tag{6.4}$$

S exists if V decreases sufficiently fast spatially or temporally.

The free evolution operator is given by

$$U_0(t'', t') = e^{-i H_0 (t''-t')/\hbar}$$

so that

$$\langle \boldsymbol{q}'' \mid U_0(t'', t') \mid \boldsymbol{q}' \rangle = \int d\boldsymbol{p}\, d\boldsymbol{p}' \langle \boldsymbol{q}'' \mid \boldsymbol{p} \rangle \langle \boldsymbol{p} \mid U_0(t'', t') \mid \boldsymbol{p}' \rangle \langle \boldsymbol{p}' \mid \boldsymbol{q}' \rangle$$
$$= \int d\boldsymbol{p}\, d\boldsymbol{p}' \frac{e^{i \boldsymbol{p}\cdot\boldsymbol{q}''/\hbar}}{(2\pi\hbar)^{d/2}} \langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle e^{-i \boldsymbol{p}'^2 (t''-t')/2m\hbar} \frac{e^{-i \boldsymbol{p}'\cdot\boldsymbol{q}'/\hbar}}{(2\pi\hbar)^{d/2}}$$

Using

$$\langle \boldsymbol{p} \mid \boldsymbol{p}' \rangle = \delta(\boldsymbol{p} - \boldsymbol{p}')$$

we have

$$\langle \boldsymbol{q}'' \mid U_0(t'', t') \mid \boldsymbol{q}' \rangle = \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^d} \exp \frac{i}{\hbar}\left[\boldsymbol{p}\cdot(\boldsymbol{q}'' - \boldsymbol{q}') - \frac{\boldsymbol{p}^2}{2m}(t'' - t')\right] \tag{6.5a}$$
$$= \left(\frac{m}{2\pi\hbar i (t'' - t')}\right)^{d/2} \exp\left[\frac{i}{\hbar} \frac{m(\boldsymbol{q}'' - \boldsymbol{q}')^2}{2(t'' - t')}\right] \tag{6.5b}$$

Eq(6.3) gives

$$\langle \mathbf{p}'' \mid \mathcal{S} \mid \mathbf{p}' \rangle = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int d\,\mathbf{p}_1\, d\,\mathbf{p}_2 \left\langle \mathbf{p}'' \right| e^{i H_0 t'' / \hbar} \left| \mathbf{p}_1 \right\rangle$$

$$\times \langle \mathbf{p}_1 \mid U(t'', t') \mid \mathbf{p}_2 \rangle \left\langle \mathbf{p}_2 \right| e^{-i H_0 t' / \hbar} \left| \mathbf{p}' \right\rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int d\,\mathbf{p}_1\, d\,\mathbf{p}_2\, e^{i E'' t'' / \hbar} \langle \mathbf{p}'' \mid \mathbf{p}_1 \rangle \langle \mathbf{p}_1 \mid U(t'', t') \mid \mathbf{p}_2 \rangle \langle \mathbf{p}_2 \mid \mathbf{p}' \rangle e^{-i E' t' / \hbar}$$

where

$$E'' = E(\mathbf{p}'') = \frac{\mathbf{p}''^2}{2m} \qquad \& \qquad E' = E(\mathbf{p}') = \frac{\mathbf{p}'^2}{2m}$$

After evaluating the delta functions, we have

$$\langle \mathbf{p}'' \mid \mathcal{S} \mid \mathbf{p}' \rangle = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{i E'' t'' / \hbar} \langle \mathbf{p}'' \mid U(t'', t') \mid \mathbf{p}' \rangle e^{-i E' t' / \hbar} \tag{6.6}$$

Defining the T-matrix $\mathcal{T}$ is related to $\mathcal{S}$ by

$$\mathcal{S} = 1 - i\,\mathcal{T}$$

$$\rightarrow \qquad \langle \mathbf{p}'' \mid \mathcal{S} \mid \mathbf{p}' \rangle = \delta(\mathbf{p}'' - \mathbf{p}') - i \langle \mathbf{p}'' \mid \mathcal{T} \mid \mathbf{p}' \rangle \tag{6.7}$$

If H is time independent, then energy is conserved so that we can write

$$\langle \mathbf{p}'' \mid \mathcal{T} \mid \mathbf{p}' \rangle = -2\pi\, \delta(E'' - E')\, T(\mathbf{p}'', \mathbf{p}') \tag{6.8}$$

Path Integrals

The Euclean (or imagiary) time path integral eq(2.19) can be converted to real time path integral by means of analytic continuation.

For a hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q}, t)$$

we get

$$\langle \mathbf{q}'' \mid U(t'', t') \mid \mathbf{q}' \rangle = \int_{\mathbf{q}(t') = q'}^{\mathbf{q}(t'') = q''} [d\,\mathbf{q}(t)] \exp\left(\frac{i}{\hbar} \mathcal{A}(\mathbf{q}) \right) \tag{6.9}$$

where

$$\mathcal{A}(\mathbf{q}) = \int_{t'}^{t''} d\,t \left(\frac{1}{2} m \dot{\mathbf{q}}^2 - V(\mathbf{q}, t) \right) = \int_{t'}^{t''} d\,t\; L \tag{6.10}$$

is the classical action.

Phase Space Formulation

Similarly, the real time version of the phase space path integral [see eq(3.9)] is

$$\langle \mathbf{q}'' \mid U(t'', t') \mid \mathbf{q}' \rangle = \int_{\mathbf{q}(t') = q'}^{\mathbf{q}(t'') = q''} [d\,\mathbf{p}(t)\, d\,\mathbf{q}(t)] \exp\left(\frac{i}{\hbar} \mathcal{A}(\mathbf{p}, \mathbf{q}) \right) \tag{6.11}$$

where

$$\mathcal{A}(\mathbf{p}, \mathbf{q}) = \int_{t'}^{t''} d\,t \left[\mathbf{p} \cdot \dot{\mathbf{q}} - H(\mathbf{p}, \mathbf{q}, t) \right] \tag{6.12}$$