6.2. Path Integral and S-Matrix: Perturbation Theory

Consider

$$H = \frac{\boldsymbol{p}^2}{2\,m} + V(\boldsymbol{x}) \tag{6.13}$$

with

$$\mathcal{A}_0(\boldsymbol{x}) = \int_{t'}^{t''} d\,t\,\frac{1}{2}\,m\,\dot{\boldsymbol{x}}^2 \qquad \mathcal{A}(\boldsymbol{x}) = \int_{t'}^{t''} d\,t\left(\frac{1}{2}\,m\,\dot{\boldsymbol{x}}^2 - V(\boldsymbol{x})\right) \tag{6.14}$$

Let

$$\begin{aligned}\langle \boldsymbol{x}'' \mid U(t'', t') \mid \boldsymbol{x}' \rangle &= \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t'')=\boldsymbol{x}''} [d\,\boldsymbol{x}(t)]\, e^{i\,\mathcal{A}(\boldsymbol{x})/\hbar} \\ &= \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t'')=\boldsymbol{x}''} [d\,\boldsymbol{x}\,(t)]\, e^{i\,\mathcal{A}_0(\boldsymbol{x})/\hbar} \sum_{l=0}^{\infty} \frac{(-i)^l}{l!}\left(\frac{1}{\hbar}\int_{t'}^{t''} d\,t\,V(\boldsymbol{x})\right)^l \\ &= \sum_i \left\langle \boldsymbol{x}'' \,\middle|\, U^{(l)}(t'', t') \,\middle|\, \boldsymbol{x}' \right\rangle \end{aligned} \tag{6.15}$$

where

$$\left\langle \boldsymbol{x}'' \,\middle|\, U^{(l)}(t'', t') \,\middle|\, \boldsymbol{x}' \right\rangle = \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t'')=\boldsymbol{x}''} [d\,\boldsymbol{x}(t)]\, e^{i\,\mathcal{A}_0(\boldsymbol{x})/\hbar}\, \frac{(-i)^l}{l!}\left(\frac{1}{\hbar}\int_{t'}^{t''} d\,t\,V(\boldsymbol{x})\right)^l \tag{6.15a}$$

Let

$$V(\boldsymbol{x}) = \int \frac{d\,\boldsymbol{k}}{(2\,\pi)^d}\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}\, \tilde{V}(\boldsymbol{k}) \tag{6.16}$$

$$\begin{aligned}\rightarrow \quad \left(\frac{1}{\hbar}\int_{t'}^{t''} d\,t\,V(\boldsymbol{x})\right)^l &= \prod_{j=1}^{l}\left\{\frac{1}{\hbar}\int_{t'}^{t''} d\,\tau_j\, V[\boldsymbol{x}(\tau_j)]\right\} \\ &= \prod_{j=1}^{l}\left\{\frac{1}{\hbar}\int_{t'}^{t''} d\,\tau_j \int \frac{d\,\boldsymbol{k}_j}{(2\,\pi)^d}\, e^{i\,\boldsymbol{k}_j\cdot\boldsymbol{x}(\tau_j)}\, \tilde{V}(\boldsymbol{k}_j)\right\} \\ &= \int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j \int\left(\prod_{j=1}^{l} \frac{d\,\boldsymbol{k}_j}{(2\,\pi)^d}\,\frac{\tilde{V}(\boldsymbol{k}_j)}{\hbar}\right) \exp\left(i\sum_{j=1}^{l} \boldsymbol{k}_j\cdot\boldsymbol{x}\,(\tau_j)\right)\end{aligned}$$

$$\begin{aligned}\therefore \quad \left\langle \boldsymbol{x}'' \,\middle|\, U^{(l)}(t'', t') \,\middle|\, \boldsymbol{x}' \right\rangle &= \frac{(-i)^l}{l!}\int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j \int\left(\prod_{j=1}^{l} \frac{d\,\boldsymbol{k}_j}{(2\,\pi)^d}\,\frac{\tilde{V}(\boldsymbol{k}_j)}{\hbar}\right) \\ &\times \int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t'')=\boldsymbol{x}''} [d\,\boldsymbol{x}(t)] \exp\left(i\left[\frac{1}{\hbar}\int_{t'}^{t''} d\,t\,\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2 + \sum_{j=1}^{l} \boldsymbol{k}_j\cdot\boldsymbol{x}\,(\tau_j)\right]\right)\end{aligned} \tag{6.17}$$

Since the integrand is symmetric in τ_j, we can set

$$t'' = \tau_{l+1} \geq \tau_l \geq \tau_{l-1} \geq \ldots \geq \tau_1 \geq \tau_0 = t' \tag{6.17a}$$

so that

$$\begin{aligned}\frac{1}{l!}\int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j\, f(\tau) &= \int_{\tau_{l-1}}^{t''} d\,\tau_l \int_{\tau_{l-2}}^{\tau_l} d\,\tau_{l-1} \ldots \int_{\tau_1}^{\tau_3} d\,\tau_2 \int_{t'}^{\tau_2} d\,\tau_1\, f(\tau) \\ &= \prod_{j=1}^{l} \int_{\tau_{j-1}}^{\tau_{j+1}} d\,\tau_j\, f(\tau)\end{aligned}$$

The last factor in (6.17) is a path integral of the generating functional type [see §2.5.1].

Writing

$$\frac{1}{\hbar}\int_{t'}^{t''} d\,t\;\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2+\sum_{j=1}^{l}\boldsymbol{k}_j\cdot\boldsymbol{x}\left(\tau_j\right)=\frac{1}{\hbar}\int_{t'}^{t''} d\,t\left[\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2+\sum_{j=1}^{l}\delta\left(t-\tau_j\right)\hbar\,\boldsymbol{k}_j\cdot\boldsymbol{x}\,(t)\right]$$

we have

$$L=\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2+\sum_{j=1}^{l}\delta\left(t-\tau_j\right)\hbar\,\boldsymbol{k}_j\cdot\boldsymbol{x}\,(t)$$

so that the Lagrange eq. is

$$m\,\ddot{\boldsymbol{x}}=\sum_{j=1}^{l}\delta\left(t-\tau_j\right)\hbar\,\boldsymbol{k}_j \tag{6.17b}$$

$$\rightarrow \qquad m\int_{t'}^{t} d\,t\,\ddot{\boldsymbol{x}}=m\left[\,\dot{\boldsymbol{x}}(t)-\dot{\boldsymbol{x}}(t')\,\right]$$

$$=\sum_{j=1}^{l}\int_{t'}^{t} d\,t\;\delta\left(t-\tau_j\right)\hbar\,\boldsymbol{k}_j$$

$$=\sum_{j=1}^{n}\hbar\,\boldsymbol{k}_j \qquad \text{where} \qquad \tau_{n+1}>t\geq\tau_n$$

$$\therefore \qquad \dot{\boldsymbol{x}}(t)=\dot{\boldsymbol{x}}(t')+\frac{1}{m}\sum_{j=1}^{n}\hbar\,\boldsymbol{k}_j$$

$$\rightarrow \qquad \dot{\boldsymbol{x}}\left(\tau_j\right)=\dot{\boldsymbol{x}}(t')+\frac{1}{m}\sum_{i=1}^{j}\hbar\,\boldsymbol{k}_i \qquad\qquad \dot{\boldsymbol{x}}\left(\tau_{j-1}\right)=\dot{\boldsymbol{x}}(t')+\frac{1}{m}\sum_{i=1}^{j-1}\hbar\,\boldsymbol{k}_i$$

$$\therefore \qquad \dot{\boldsymbol{x}}\left(\tau_j\right)=\dot{\boldsymbol{x}}\left(\tau_{j-1}\right)+\frac{1}{m}\,\hbar\,\boldsymbol{k}_j \tag{6.17c}$$

Since

$$\dot{\boldsymbol{x}}(t)=\dot{\boldsymbol{x}}\left(\tau_j\right) \qquad \text{for} \qquad \tau_{j+1}\geq t\geq\tau_j$$

the particle is basically in free motion except for bumps at τ_j's.

Thus we can break up the path integral into $l+1$ segments:

$$\mathcal{P}_0=\int_{\boldsymbol{x}(t')=\boldsymbol{x}'}^{\boldsymbol{x}(t'')=\boldsymbol{x}''}[d\,\boldsymbol{x}(t)]\exp\left(i\left[\frac{1}{\hbar}\int_{t'}^{t''} d\,t\;\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2\right]\right)$$

$$=\int\prod_{j=1}^{l}\left(d\,\boldsymbol{x}_j\int_{\boldsymbol{x}(\tau_{j-1})=\boldsymbol{x}_{j-1}}^{\boldsymbol{x}(\tau_j)=\boldsymbol{x}_j}[d\,\boldsymbol{x}(t)]\right)\exp\left(i\left[\frac{1}{\hbar}\sum_{j=1}^{l+1}\int_{\tau_{j-1}}^{\tau_j} d\,t\;\frac{1}{2}\,m\,\dot{\boldsymbol{x}}(t)^2\right]\right)$$

where

$$\boldsymbol{x}\left(\tau_j\right)=\boldsymbol{x}_j \qquad \boldsymbol{x}_{l+1}=\boldsymbol{x}'' \qquad \& \qquad \boldsymbol{x}_0=\boldsymbol{x}'$$

Setting

$$\boldsymbol{p}_j=m\,\dot{\boldsymbol{x}}_j \qquad \boldsymbol{p}_{l+1}=m\,\dot{\boldsymbol{x}}_{l+1} \qquad \& \qquad \boldsymbol{p}_0=m\,\dot{\boldsymbol{x}}_0$$

we have

$$\mathcal{P}_0=\int\left(\prod_{j=1}^{l}d\,\boldsymbol{x}_j\right)\prod_{j=1}^{l+1}\left\langle\,\boldsymbol{x}_j\;\middle|\;U_0\left(\tau_j,\,\tau_{j-1}\right)\;\middle|\;\boldsymbol{x}_{j-1}\,\right\rangle$$

$$=\int\left(\prod_{j=1}^{l}d\,\boldsymbol{x}_j\right)\prod_{j=1}^{l+1}\left\langle\,\boldsymbol{x}_j\;\middle|\;e^{-i\,\hat{\boldsymbol{p}}^2(\tau_j-\tau_{j-1})/2\,m\,\hbar}\;\middle|\;\boldsymbol{x}_{j-1}\,\right\rangle$$

$$= \int \left(\prod_{j=1}^{l} d\,\boldsymbol{x}_j \right) \prod_{j=1}^{l+1} \int d\,\boldsymbol{p}'\, d\,\boldsymbol{p}_{j-1} \langle \boldsymbol{x}_j \mid \boldsymbol{p}' \rangle \left\langle \boldsymbol{p}' \left| e^{-i\,\hat{\boldsymbol{p}}^2 (\tau_j - \tau_{j-1}) / 2 m \hbar} \right| \boldsymbol{p}_{j-1} \right\rangle \langle \boldsymbol{p}_{j-1} \mid \boldsymbol{x}_{j-1} \rangle$$

$$= \int \left(\prod_{j=1}^{l} d\,\boldsymbol{x}_j \right) \prod_{j=1}^{l+1} \int d\,\boldsymbol{p}'\; d\,\boldsymbol{p}_{j-1} \frac{e^{i\,\boldsymbol{p}' \cdot \boldsymbol{x}_j / \hbar}}{(2\pi\hbar)^{d/2}} e^{i\,\boldsymbol{p}_{j-1}^2 (\tau_j - \tau_{j-1}) / 2 m \hbar} \langle \boldsymbol{p}' \mid \boldsymbol{p}_{j-1} \rangle \frac{e^{-i\,\boldsymbol{p}_{j-1} \cdot \boldsymbol{x}_{j-1} / \hbar}}{(2\pi\hbar)^{d/2}}$$

$$= \int \left(\prod_{j=1}^{l} d\,\boldsymbol{x}_j \right) \prod_{j=1}^{l+1} \int \frac{d\,\boldsymbol{p}_{j-1}}{(2\pi\hbar)^d} \exp\left\{ \frac{i}{\hbar} \left(-\frac{\boldsymbol{p}_{j-1}^2}{2m} (\tau_j - \tau_{j-1}) + \boldsymbol{p}_{j-1} \cdot (\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \right) \right\}$$

Eq(6.17) thus becomes

$$\langle \boldsymbol{x}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{x}' \rangle = (-i)^l \int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j \int \left(\prod_{j=1}^{l} d\,\boldsymbol{x}_j \frac{d\,\boldsymbol{k}_j}{(2\pi)^d} \frac{\tilde{V}(\boldsymbol{k}_j)}{\hbar} \right) \int \left(\prod_{j=1}^{l+1} \frac{d\,\boldsymbol{p}_{j-1}}{(2\pi\hbar)^d} \right)$$

$$\times \exp\left\{ \frac{i}{\hbar} \left(\sum_{j=1}^{l+1} \left[-\frac{\boldsymbol{p}_{j-1}^2}{2m} (\tau_j - \tau_{j-1}) + \boldsymbol{p}_{j-1} \cdot (\boldsymbol{x}_j - \boldsymbol{x}_{j-1}) \right] + \sum_{j=1}^{l} \hbar\,\boldsymbol{k}_j \cdot \boldsymbol{x}_j \right) \right\}$$

with the constraints on τ_j given by eq(6.17a).

Collecting all terms involving a given $\boldsymbol{x}_j$ in the exponential of eq(6.17g), we get

$$\int d\,\boldsymbol{x}_j \exp\left[i \left(\frac{\boldsymbol{p}_{j-1} - \boldsymbol{p}_j}{\hbar} + \boldsymbol{k}_j \right) \cdot \boldsymbol{x}_j \right] = (2\pi)^d\, \delta\left(\frac{\boldsymbol{p}_{j-1} - \boldsymbol{p}_j}{\hbar} + \boldsymbol{k}_j \right) \qquad j = 1, \ldots, l$$

Performing the $\boldsymbol{k}_j$ integrals then gives

$$\langle \boldsymbol{x}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{x}' \rangle = (-i)^l \int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j \int \left(\prod_{j=1}^{l+1} \frac{d\,\boldsymbol{p}_{j-1}}{(2\pi\hbar)^d} \right) \left(\prod_{j=1}^{l} \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{\hbar} \right)$$

$$\times \exp\left\{ \frac{i}{\hbar} \left(-\sum_{j=1}^{l+1} \frac{\boldsymbol{p}_{j-1}^2}{2m} (\tau_j - \tau_{j-1}) + \boldsymbol{p}_l \cdot \boldsymbol{x}'' - \boldsymbol{p}_0 \cdot \boldsymbol{x}' \right) \right\}$$

Consider now the matrix elements with respect to the momentum eigenstates

$$\langle \boldsymbol{p}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{p}' \rangle = \int d\,\boldsymbol{x}''\, d\,\boldsymbol{x}' \langle \boldsymbol{p}'' \mid \boldsymbol{x}'' \rangle \langle \boldsymbol{x}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{x}' \rangle \langle \boldsymbol{x}' \mid \boldsymbol{p}' \rangle$$

$$= \int d\,\boldsymbol{x}''\, d\,\boldsymbol{x}' \frac{e^{-i\,\boldsymbol{p}'' \cdot \boldsymbol{x}'' / \hbar}}{(2\pi\hbar)^{d/2}} \langle \boldsymbol{x}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{x}' \rangle \frac{e^{i\,\boldsymbol{p}' \cdot \boldsymbol{x}' / \hbar}}{(2\pi\hbar)^{d/2}}$$

The $\boldsymbol{x}''$ & $\boldsymbol{x}'$ integrals give

$$\int d\,\boldsymbol{x}''\, e^{i (\boldsymbol{p}_l - \boldsymbol{p}'') \cdot \boldsymbol{x}'' / \hbar} = (2\pi\hbar)^d\, \delta(\boldsymbol{p}_l - \boldsymbol{p}'')$$

$$\int d\,\boldsymbol{x}'\, e^{i (\boldsymbol{p}' - \boldsymbol{p}_0) \cdot \boldsymbol{x}' / \hbar} = (2\pi\hbar)^d\, \delta(\boldsymbol{p}' - \boldsymbol{p}_0)$$

Performing the $\boldsymbol{p}_l$ & $\boldsymbol{p}_0$ integrals then gives

$$\langle \boldsymbol{p}'' \mid U^{(l)}(t'', t') \mid \boldsymbol{p}' \rangle = (2\pi\hbar)^d (-i)^l \int_{t'}^{t''} \prod_{j=1}^{l} d\,\tau_j \int \left(\prod_{j=2}^{l} \frac{d\,\boldsymbol{p}_{j-1}}{(2\pi\hbar)^d} \right) \left(\prod_{j=1}^{l} \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{\hbar} \right)$$

$$\times \exp\left(-\frac{i}{\hbar} \sum_{j=1}^{l+1} \frac{\boldsymbol{p}_{j-1}^2}{2m} (\tau_j - \tau_{j-1}) \right)$$

$$= (2\pi\hbar)^d (-i)^l \int_{t'}^{t''} \prod_{j=1}^{l} d\tau_j \int \left(\prod_{j=1}^{l-1} \frac{d\boldsymbol{p}_j}{(2\pi\hbar)^d} \right) \left(\prod_{j=1}^{l} \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{\hbar} \right) \times \exp\left(-\frac{i}{\hbar} \sum_{j=0}^{l} \frac{\boldsymbol{p}_j^2}{2m} (\tau_{j+1} - \tau_j) \right)$$

with the B.C.'s

$$\boldsymbol{p}_l = \boldsymbol{p}'' \quad \& \quad \boldsymbol{p}' = \boldsymbol{p}_0 \tag{6.18a}$$

The S-matrix is

$$\langle \boldsymbol{p}'' | S^{(l)} | \boldsymbol{p}' \rangle = \lim_{\substack{t'' \to \infty \\ t' \to -\infty}} e^{i E'' t'' / \hbar} \langle \boldsymbol{p}'' | U^{(l)}(t'', t') | \boldsymbol{p}' \rangle e^{-i E' t' / \hbar} \tag{6.18b}$$

with

$$E'' = \frac{\boldsymbol{p}_l^2}{2m} \quad \& \quad E' = \frac{\boldsymbol{p}_0^2}{2m}$$

so that the τ_{l+1} & τ_0 terms in the exponential are cancelled out. This effect can be emulated by setting

$$\tau_{l+1} = \tau_0 = 0$$

(6.18c)

whereupon

$$\langle \boldsymbol{p}'' | S^{(l)} | \boldsymbol{p}' \rangle = (2\pi\hbar)^d (-i)^l \int_{-\infty}^{\infty} \prod_{j=1}^{l} d\tau_j \int \left(\prod_{j=1}^{l-1} \frac{d\boldsymbol{p}_j}{(2\pi\hbar)^d} \right) \left(\prod_{j=1}^{l} \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{\hbar} \right) \times \exp\left(-\frac{i}{\hbar} \sum_{j=0}^{l} \frac{\boldsymbol{p}_j^2}{2m} (\tau_{j+1} - \tau_j) \right) \tag{6.18}$$

Collecting all terms involving a given τ_j in the exponential, we get

$$\mathcal{I}_j = \int_{\tau_{j-1}}^{\tau_{j+1}} d\tau_j \exp\left(-\frac{i}{\hbar} \frac{(\boldsymbol{p}_{j-1}^2 - \boldsymbol{p}_j^2)}{2m} \tau_j \right)$$

$$= \int_{\tau_{j-1}}^{\tau_{j+1}} d\tau_j \exp\left(\frac{i}{\hbar} [E(\boldsymbol{p}_j) - E(\boldsymbol{p}_{j-1})] \tau_j \right) \qquad \text{where} \quad E(\boldsymbol{p}) = \frac{\boldsymbol{p}^2}{2m}$$

Since τ_{j+1} & τ_{j-1} can assume any values satisfying the only restriction $\tau_{j+1} \ge \tau_{j-1}$, we can write

$$\mathcal{I}_j = \int_0^{\infty} d\tau_j \exp\left(\frac{i}{\hbar} [E'' - E(\boldsymbol{p}_{j-1})] \tau_j \right) \qquad \forall\ j = 2, \ldots, l$$

where we've incorporated the fact that $\tau_j = \infty$ is possible only if $\tau_k = \infty\ \forall\ k > j$.

To make the exponential vanish at $\tau_j = \infty$, we set, with $\epsilon \to 0^+$,

$$\mathcal{I}_j = \int_0^{\infty} d\tau_j \exp\left(\frac{i}{\hbar} [E'' - E(\boldsymbol{p}_{j-1}) + i\epsilon] \tau_j \right)$$

$$= \frac{i\hbar}{E'' - E(\boldsymbol{p}_{j-1}) + i\epsilon} \qquad \forall\ j = 2, \ldots, l$$

Since $\tau_0 = -\infty$, the case $j = 1$ gives

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} d\tau_j \exp\left(\frac{i}{\hbar} (E'' - E') \tau_j \right) = 2\pi\hbar\, \delta(E'' - E')$$

Thus, eq(6.18) beocmes

$$\langle \boldsymbol{p}'' | S^{(l)} | \boldsymbol{p}' \rangle = (2\pi\hbar)^d\, 2\pi\hbar\, \delta(E'' - E') (-i)^l \times \int \left(\prod_{j=1}^{l-1} \frac{d\boldsymbol{p}_j}{(2\pi\hbar)^d} \right) \left(\prod_{j=1}^{l} \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{\hbar} \right) \left(\prod_{j=2}^{l} \frac{i\hbar}{E'' - E(\boldsymbol{p}_{j-1}) + i\epsilon} \right)$$

$$= -i\,(2\,\pi\,\hbar)^d\,2\,\pi\,\delta(E'' - E')$$
$$\times \int \left(\prod_{j=1}^{l-1} \frac{d\,\boldsymbol{p}_j}{(2\,\pi\,\hbar)^d} \right) \left(\prod_{j=1}^{l} \tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1}) \right) \left(\prod_{j=1}^{l-1} \frac{1}{E'' - E(\boldsymbol{p}_j) + i\,\epsilon} \right)$$
$$= -i\,(2\,\pi\,\hbar)^d\,2\,\pi\,\delta(E'' - E') \tag{6.18d}$$
$$\times \int \tilde{V}(\boldsymbol{p}'' - \boldsymbol{p}_{l-1}) \prod_{j=1}^{l-1} \frac{d\,\boldsymbol{p}_j}{(2\,\pi\,\hbar)^d}\, \frac{\tilde{V}(\boldsymbol{p}_j - \boldsymbol{p}_{j-1})}{E'' - E(\boldsymbol{p}_j) + i\,\epsilon}$$

Note: The $(2\,\pi\,\hbar)^d$ factor would be absent if we used normalized plane waves in eq(6.18b).

Eq(6.18b) can also be obtained by solving the Lippman-Schwinger (integral) eq. by iteration.

The equivalent operator eq. is usually written in terms of the T-matrix [see e.g., §4.1 of E.N.Economou, “Green’s Functions in Quantum Physics”] :

$$T(E) = V - V\,G_0(E)\,T(E) \qquad G_0(E) = \frac{1}{H_0 - E}$$

with

$$T(\boldsymbol{p}'', \boldsymbol{p}') = \langle \boldsymbol{p}'' \mid T(E + i\,\epsilon) \mid \boldsymbol{p}' \rangle \qquad E = \frac{\boldsymbol{p}'^2}{2\,m} = \frac{\boldsymbol{p}''^2}{2\,m}$$