6.4. S-Matrix and Holomorphic Formalism

This is the real time version of the holomorphic formalism discussed in §5.1.

Caution: Our choice of using $\langle z'' \mid U(t'', t') \mid z' \rangle$ instead of Zinn-Justin's $\langle z'' \mid U(t'', t') \mid \bar{z}' \rangle$ persists here but won't be highlighted.

6.4.1. Path integrals

Taking the analytic continuation $\tau \to i\,t$, where τ is the imaginary time, we get from eqs(5.34p & 5.35) of §5.2.1,

$$U(z'', z'; t'', t') = \langle z'' \mid U(t'', t') \mid z' \rangle$$

$$= \int [d\,\bar{z}(t)\, d\,z(t)]\; e^{i\,\mathcal{A}(z,\bar{z})}$$

$$\mathcal{A} = i\,S = -i\,z(\tau')\,\bar{z}(\tau') + i\int_{\tau'}^{\tau''} d\,\tau \left(-\bar{z}\frac{d\,z}{d\,\tau} + \frac{1}{\hbar}\,h \right)$$

$$= -i\,z(t')\,\bar{z}(t') - \int_{t'}^{t''} d\,t \left(i\,\bar{z}\,\dot{z} + \frac{1}{\hbar}\,h \right) \qquad (6.38a)$$

with the B.C.

$$z(t'') = z'' \qquad\qquad z(t') = z'$$

Let [see eq(5.3)]

$$H_0 = \hbar\,\omega\,a^+\,a = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\,\hat{q}^2 - \frac{1}{2}\hbar\,\omega$$

$$= \hbar\,\omega\,z\,\frac{\partial}{\partial\,z}$$

then eq(5.14p) gives

$$U_0(z'', z'; t'', t') = \langle z'' \mid U_0(t'', t') \mid z' \rangle$$

$$= \exp\left(e^{-i\,\omega\,(t''-t')}\, z''\,\bar{z}'\right)$$

The S-matrix definition eq(6.3) can be written as

$$\mathcal{S} = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{i\,H_0\,t''}\, U(t'', t')\, e^{-i\,H_0\,t'}$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} U_0(0, t'')\, U(t'', t')\, U_0(t', 0) \qquad (6.38b)$$

Using the completeness relation eq(5.8b) of §5.1:

$$\int \frac{d\,\bar{z}\,d\,z}{2\,\pi\,i}\, e^{-z\,\bar{z}} \mid z \rangle\langle z \mid = I$$

we have

$$\mathcal{S}(z'', z') = \langle z'' \mid \mathcal{S} \mid z' \rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \langle z'' \mid U_0(0, t'')\, U(t'', t')\, U_0(t', 0) \mid z' \rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int \frac{d\,\bar{z}_1\,d\,z_1}{2\,\pi\,i}\, e^{-z_1\,\bar{z}_1} \int \frac{d\,\bar{z}_2\,d\,z_2}{2\,\pi\,i}\, e^{-z_2\,\bar{z}_2}$$

$$\times \langle z'' \mid U_0(0, t'') \mid z_1 \rangle \langle z_1 \mid U(t'', t') \mid z_2 \rangle \langle z_2 \mid U_0(t', 0) \mid z' \rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int \frac{d\bar{z}_1\, dz_1}{2\pi i} e^{-z_1 \bar{z}_1} \int \frac{d\bar{z}_2\, dz_2}{2\pi i} e^{-z_2 \bar{z}_2}$$

$$\times \exp\left(e^{i\omega t''} z'' \bar{z}_1\right) \langle z_1 \mid U(t'', t') \mid z_2 \rangle \exp\left(e^{-i\omega t'} z_2 \bar{z}'\right)$$

Using [see eq(5.10a)]

$$\int \frac{d\bar{z}_1}{2\pi i} \exp\left[-\left(z_1 - e^{i\omega t''} z''\right) \bar{z}_1\right] = \delta\left(z_1 - e^{i\omega t''} z''\right)$$

$$\int \frac{d\bar{z}_2}{2\pi i} \exp(-z_2 \bar{z}_2) = \delta(z_2)$$

we get the alarming result

$$\mathcal{S}(z'', z') = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \left\langle e^{i\omega t''} z'' \,\middle|\, U(t'', t') \,\middle|\, 0 \right\rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} U\left(e^{i\omega t''} z'', 0\,;\, t'', t'\right) \qquad (6.38c)$$

that is independent of z'.

A more conventional form of the S-matrix can be obtained as follows

$$\mathcal{S}(z'', z') = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \int \frac{d\bar{z}_1\, dz_1}{2\pi i} e^{-z_1 \bar{z}_1} \exp\left(e^{i\omega t''} z'' \bar{z}_1\right)$$

$$\times \langle z_1 \mid U(t'', t')\, U_0(t', 0) \mid z' \rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \left\langle e^{i\omega t''} z'' \,\middle|\, U(t'', t')\, U_0(t', 0) \,\middle|\, z' \right\rangle$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \overline{\left\langle z' \,\middle|\, U_0^{+}(t', 0)\, U^{+}(t'', t') \,\middle|\, e^{i\omega t''} z'' \right\rangle}$$

Using

$$\left\langle z' \mid U_0^{+}(t', 0) \mid z_1 \right\rangle = \overline{\langle z_1 \mid U_0(t', 0) \mid z' \rangle}$$

$$= \overline{\exp\left(e^{-i\omega t'} z_1 \bar{z}'\right)}$$

$$= \exp\left(e^{i\omega t'} \bar{z}_1 z'\right)$$

we have

$$\mathcal{S}(z'', z') = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \overline{\int \frac{d\bar{z}_1\, dz_1}{2\pi i} e^{-z_1 \bar{z}_1} \exp\left(e^{i\omega t'} z' \bar{z}_1\right)}$$

$$\times \overline{\left\langle z_1 \,\middle|\, U^{+}(t'', t') \,\middle|\, e^{i\omega t''} z'' \right\rangle}$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \overline{\left\langle e^{i\omega t'} z' \,\middle|\, U^{+}(t'', t') \,\middle|\, e^{i\omega t''} z'' \right\rangle}$$

$$= \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \left\langle e^{i\omega t''} z'' \,\middle|\, U(t'', t') \,\middle|\, e^{i\omega t'} z' \right\rangle \qquad (6.38)$$

From (ii) of §“Remarks” in §5.1, we have

$$\mathcal{S}(z'', z') = \sum_{m,n} S_{mn} \frac{z''^{m}}{\sqrt{m!}} \frac{\bar{z}'^{n}}{\sqrt{n!}} \qquad (6.38c)$$

where

$$S_{mn} = \langle m \mid S \mid n \rangle \qquad \langle z \mid m \rangle = \frac{z^m}{\sqrt{m!}}$$

The classical solution for H_0 is [see eq(a) below eq(5.28a) in §5.2]

$$z(t) = z\, e^{i\omega t}$$

which should be a good approximation for H at large times provided $H - H_0 \to 0$ at large times or distances.

6.4.2. Time-Dependent Force

Let [see eq(5.26t) of §5.2]

$$H = H_0 - \hbar\left[\,\bar{b}(t)\, z(t) + b(t)\, \bar{z}(t)\,\right]$$
$$= \hbar\left(\omega z \bar{z} - \bar{b} z - b \bar{z}\right)$$

where

$$b(t),\ \bar{b}(t) \to 0 \quad \text{for} \qquad t \to \pm\infty$$

From eqs(5.28a & 6.38a), we have

$$\mathcal{A} = -i\, z(t')\, \bar{z}(t') - \int_{t'}^{t''} dt \left(i \bar{z} \dot{z} + \omega z \bar{z} - \bar{b} z - b \bar{z}\right)$$

Following the derivation of §5.2, we have:

$$L = i \bar{z} \dot{z} + \omega z \bar{z} - \bar{b} z - b \bar{z}$$

$$\to \qquad \frac{\partial L}{\partial \dot{z}} = i \bar{z} \qquad\qquad \frac{\partial L}{\partial z} = \omega \bar{z} - \bar{b}$$

$$i \dot{\bar{z}} - \omega \bar{z} = -\bar{b}$$

with the periodic B.C.

$$\bar{z}(t'') = \bar{z}(t') \tag{6.39a}$$

The homogenous solution is

$$\bar{z}_h(t) = c\, e^{-i\omega t}$$

Eq. for the green function is

$$i \dot{\Delta} - \omega \Delta = \delta(t)$$

with discontinuity

$$i\Delta(\varepsilon) - i\Delta(-\varepsilon) = 1 \qquad \text{as} \qquad \varepsilon \to 0_+$$

& periodic B.C.

$$\Delta(t'') = \Delta(t')$$

$$\Delta(t) = \begin{cases} c\, e^{-i\omega t} & \text{for } t'' > t > 0 \\ c'\, e^{-i\omega t} & \text{for } 0 < t < t' \end{cases}$$

$$\therefore \qquad i c - i c' = 1$$
$$c\, e^{-i\omega t''} = c'\, e^{-i\omega t'}$$

$$\to \qquad c = \frac{i}{e^{-i\omega(t''-t')} - 1} \qquad\qquad c' = \frac{i\, e^{-i\omega(t''-t')}}{e^{-i\omega(t''-t')} - 1}$$

$$\therefore \qquad \Delta(t) = \frac{i\, e^{-i\omega t}}{e^{-i\omega(t''-t')} - 1} \begin{cases} 1 & \text{for } t'' > t > 0 \\ e^{-i\omega(t''-t')} & \text{for } 0 < t < t' \end{cases}$$

Using

$$\frac{1}{1 - e^{-i\omega T}} = 1 + \frac{e^{-i\omega T}}{1 - e^{-i\omega T}} \qquad \text{where} \qquad T = t'' - t'$$

we have

$$\Delta(t) = -i\, e^{-i\,\omega\, t} \left(\theta(t) + \frac{e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}} \right)$$

$$= -i\, e^{-i\,\omega\,t}\, \theta(t) \qquad \text{for} \qquad T \to \infty$$

where we've set $\omega \to \omega - i\,0_+$ so that

$$\lim_{T\to\infty} e^{-i\,(\omega - i\,0_+)\,T} = 0$$

$$\rightarrow \qquad \overline{z}_c(t) = \overline{z}_h\,(t) - \int_{t'}^{t''} d\,u\,\Delta(t-u)\,\overline{b}(u)$$

$$= c\, e^{-i\,\omega\,t} + i \int_{t'}^{t''} d\,u\, e^{-i\,\omega\,(t-u)} \left(\theta(t-u) + \frac{e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}} \right) \overline{b}(u)$$

$$= c\, e^{-i\,\omega\,t} + i \int_{-\infty}^{\infty} d\,u\, e^{-i\,\omega\,(t-u)}\, \theta(t-u)\,\overline{b}(u) \qquad \text{for} \qquad T \to \infty$$

Using

$$\theta(t' - u) = 0$$

we have

$$\overline{z}_c(t') = e^{-i\,\omega\,t'}\,\overline{z}' = c\, e^{-i\,\omega\,t'} + i\, \frac{e^{-i\,\omega\,t'}\, e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}} \int_{t'}^{t''} d\,u\, e^{i\,\omega\,u}\,\overline{b}(u)$$

$$\rightarrow \qquad c = \overline{z}' - i\, \frac{e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}} \int_{t'}^{t''} d\,u\, e^{i\,\omega\,u}\,\overline{b}(u)$$

Using

$$\theta(t'' - u) = 1$$

we have

$$\overline{z}_c(t'') = e^{-i\,\omega\,t''}\,\overline{z}'' = c\, e^{-i\,\omega\,t''} + i\, \frac{e^{-i\,\omega\,t''}}{1 - e^{-i\,\omega\,T}} \int_{t'}^{t''} d\,u\, e^{i\,\omega\,u}\,\overline{b}(u)$$

$$\rightarrow \qquad c = \overline{z}'' - i\, \frac{1}{1 - e^{-i\,\omega\,T}} \int_{t'}^{t''} d\,u\, e^{i\,\omega\,u}\,\overline{b}(u)$$

$$\therefore \qquad c = \frac{\overline{z}' - \overline{z}''\, e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}}$$

$$= \overline{z}' \qquad \text{for} \quad T \to \infty \quad \& \quad \omega \to \omega - i\,0_+$$

$$\rightarrow \qquad \overline{z}_c(t) = \frac{\overline{z}' - \overline{z}''\, e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}}\, e^{-i\,\omega\,t} + i \int_{t'}^{t''} d\,u\, e^{-i\,\omega\,(t-u)} \left(\theta(t-u) + \frac{e^{-i\,\omega\,T}}{1 - e^{-i\,\omega\,T}} \right) \overline{b}(u)$$

$$= \overline{z}'\, e^{-i\,\omega\,t} + i \int_{-\infty}^{\infty} d\,u\, e^{-i\,\omega\,(t-u)}\, \theta(t-u)\; \overline{b}(u) \qquad \text{for} \quad T \to \infty$$

Using the classical approximation [see §5.2], we set

$$\mathcal{A}\,(z, \overline{z}_c) = -i\, z(t')\, \overline{z}_c\,(t') - \int_{t'}^{t''} d\,t \left(i\, \overline{z}_c\, \dot{z} + \omega\, z\, \overline{z}_c - \overline{b}\, z - b\, \overline{z}_c \right)$$

$$= -i\, z\,(t'')\, \overline{z}_c(t'') - \int_{t'}^{t''} d\,t \left[\left(-i\, \dot{\overline{z}}_c + \omega\; \overline{z}_c - \overline{b} \right) z - b\, \overline{z}_c \right)$$

$$= -i\, z''\, \overline{z}'' + \int_{t'}^{t''} d\,t\, b\, \overline{z}_c$$

For $T \to \infty$, we have

$$\mathcal{A}(z, \bar{z}_c) = -i\, z''\bar{z}'' + \int_{-\infty}^{\infty} dt\, b(t)\, \bar{z}'\, e^{-i\omega t}$$
$$+ i \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du\, b(t)\, e^{-i\omega(t-u)}\, \theta(t-u)\, \bar{b}(u)$$

Eq(6.38) thus becomes

$$\begin{aligned} \mathcal{S}(z'', z') &= C \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} \exp[i\,\mathcal{A}(z, \bar{z}_c)] \\ &= C \exp\Big(z''\bar{z}'' + i \int_{-\infty}^{\infty} dt\, b(t)\, \bar{z}'\, e^{-i\omega t} \\ &\quad - \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du\, b(t)\, e^{-i\omega(t-u)}\, \theta(t-u)\, \bar{b}(u) \Big) \end{aligned} \tag{6.39}$$

where C is such that

$$\mathcal{S}(z'', z')|_{b=\bar{b}=0} = \mathcal{S}_0(z'', z')$$

Let

$$b(t) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{i\nu t}\, \tilde{b}(\nu) \qquad \tilde{b}(\nu) = \int_{-\infty}^{\infty} dt\, e^{-i\nu t}\, b(t)$$

$$\to \quad \int_{-\infty}^{\infty} dt\, b(t)\, e^{-i\omega t} = \tilde{b}(\omega)$$

$$\begin{aligned} &\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du\, b(t)\, e^{-i\omega(t-u)}\, \theta(t-u)\, \bar{b}(u) \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} du\, b(t)\, e^{-i\omega(t-u)}\, \bar{b}(u) \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{t} du \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu'}{2\pi} e^{i\nu t - i\nu' u}\, \tilde{b}(\nu)\, \bar{\tilde{b}}(\nu')\, e^{-i\omega(t-u)} \\ &= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu'}{2\pi} e^{i(\nu-\omega)t}\, \tilde{b}(\nu)\, \bar{\tilde{b}}(\nu')\, \frac{e^{i(\omega-\nu')t}}{i(\omega - \nu' - i\varepsilon)} \\ &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{-\infty}^{\infty} d\nu'\, \delta(\nu - \nu')\, \tilde{b}(\nu)\, \bar{\tilde{b}}(\nu')\, \frac{1}{i(\omega - \nu' - i\varepsilon)} \\ &= \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \left| \tilde{b}(\nu) \right|^2 \frac{1}{i(\omega - \nu - i\varepsilon)} \end{aligned}$$

where the $i\varepsilon$ term, with $\varepsilon \to 0_+$, was added to make the u integral finite.

Eq(6.39) thus becomes

$$\mathcal{S}(z'', z') = C \exp\left(z''\bar{z}'' + i\,\tilde{b}(\omega)\,\bar{z}' - \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \left| \tilde{b}(\nu) \right|^2 \frac{i}{\nu - \omega + i\varepsilon} \right) \tag{6.40}$$

Coupling to Position Only

Let b be real, then

$$\begin{aligned} \bar{b}\, z + b\, \bar{z} &= b\,(z + \bar{z}) \\ &= b\,(a^{+} + a) && \text{[Eqs(5.4 \& 5.11) used]} \\ &= b \sqrt{\frac{2\,\omega}{\hbar}}\, q && \text{[Eq(5.2) used]} \end{aligned}$$

i.e., the perturbation is coupled to the position $q(t)$ only.

Using

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du\, b(t)\, e^{-i\omega(t-u)}\, \theta(t-u)\, b(u)$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} d\,u\, b(t)\, e^{-i\,\omega\,(t-u)}\,\theta(t-u)\, b(u) \right.$$
$$\left. + \int_{-\infty}^{\infty} d\,u \int_{-\infty}^{\infty} d\,t\, b(u)\, e^{-i\,\omega\,(u-t)}\,\theta(u-t)\, b(t) \right)$$
$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} d\,t \int_{-\infty}^{t} d\,u\, b(t)\, e^{-i\,\omega\,(t-u)}\, b(u) \right.$$
$$\left. + \int_{-\infty}^{\infty} d\,t \int_{t}^{\infty} d\,u\, b(u)\, e^{-i\,\omega\,(u-t)}\, b(t) \right)$$
$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} d\,t \int_{-\infty}^{t} d\,u\, b(t)\, e^{-i\,\omega\,|\,t-u\,|}\, b(u) \right.$$
$$\left. + \int_{-\infty}^{\infty} d\,t \int_{t}^{\infty} d\,u\, b(u)\, e^{-i\,\omega\,|\,t-u\,|}\, b(t) \right)$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} d\,u\, b(t)\, e^{-i\,\omega\,|\,t-u\,|}\, b(u)$$

eq(6.39) becomes

$$\mathcal{S}(z'', z') = \exp\left(z''\,\bar{z}'' + i \int_{-\infty}^{\infty} d\,t\, b(t)\, \bar{z}'\, e^{-i\,\omega\,t} - \frac{1}{2} \int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} d\,u\, b(t)\, e^{-i\,\omega\,|\,t-u\,|}\, b(u) \right) \tag{6.41}$$

b real $\quad\rightarrow\quad \overline{\tilde{b}}(\nu) = \tilde{b}(-\nu)$

$\therefore \quad \left|\, \tilde{b}(\nu) \right|^2 = \tilde{b}(\nu)\, \tilde{b}(-\nu)$ is invariant under the transform $\nu \to -\nu$.

$$\rightarrow \quad \int_{-\infty}^{\infty} \frac{d\,\nu}{2\,\pi} \left| \tilde{b}(\nu) \right|^2 \frac{i}{\nu - \omega + i\,\varepsilon}$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\,\nu}{2\,\pi} \left| \tilde{b}(\nu) \right|^2 \left(\frac{i}{\nu - \omega + i\,\varepsilon} - \frac{i}{\nu + \omega - i\,\varepsilon} \right)$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\,\nu}{2\,\pi} \left| \tilde{b}(\nu) \right|^2 \frac{2\,\omega\, i}{\nu^2 - \omega^2 + 2\, i\,\omega\,\varepsilon}$$

Eq(6.40) thus becomes

$$\mathcal{S}(z'', z') = \exp\left(z''\,\bar{z}'' + i\,\tilde{b}(\omega)\,\bar{z}' - \int_{-\infty}^{\infty} \frac{d\,\nu}{2\,\pi} \left| \tilde{b}(\nu) \right|^2 \frac{\omega\, i}{\nu^2 - \omega^2 + 2\, i\,\varepsilon} \right) \tag{6.42}$$

In terms of the $(p,\, q)$ variables, the equivalent real-time path integral is

$$\mathcal{Z}(b) = \int [d\,q(t)]\, e^{i\,\mathcal{A}(q,b)}$$
$$\mathcal{A}(q,\, b) = \int_{-\infty}^{\infty} d\,t \left[\frac{1}{2}\,\dot{q}^2 - \frac{1}{2}\,\omega^2\, q^2 + \sqrt{\frac{2\,\omega}{\hbar}}\; b\, q \right] \tag{6.42a}$$

Following the procedure used in §2.5 for the imaginary time version, we obtain, by replacing t with $i\,t$ in eqs(2.46-7),

$$\Delta(t) = \frac{1}{2\,\omega}\, e^{-i\,\omega\,|t|}$$
$$\mathcal{Z}(b) = \mathcal{Z}_0 \exp\left\{ -\frac{1}{2} \int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} d\,u\, b(t)\, e^{-i\,\omega\,|\,t-u\,|}\, b(u) \right\} \tag{6.42b}$$

where $\omega \to \omega - i\,0_+$ is understood to keep everything well-behaved as $|\,t\,| \to \infty$.

Eq(6.42b) explicitly reproduces the quadratic b term in eq(6.41). The linear b term can be obtained by adding to $\mathcal{A}(q, b)$ a term that does not affect the dynamics:

$$b(t)\,\overline{z}'\,e^{-i\omega t}$$

Eq(6.42a) can then be written as

$$\mathcal{A}(q, b) = \int_{-\infty}^{\infty} dt \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + \sqrt{\frac{2\omega}{\hbar}}\; b\,(q + q_0) \right] \qquad (6.42c)$$

where

$$\sqrt{\frac{2\omega}{\hbar}}\; q_0 = \overline{z}'\,e^{-i\omega t} \qquad \to \qquad q_0 = \sqrt{\frac{\hbar}{2\omega}}\;\overline{z}'\,e^{-i\omega t} \qquad (6.43)$$

Using eq(6.42c) to calculate $\mathcal{Z}(b)$, we have

$$S(z'', z') = C\exp(z''\,\overline{z}'')\,\frac{\mathcal{Z}(b)}{\mathcal{Z}_0}$$

Setting

$$Q(t) = q(t) + q_0(t)$$

we can write eq(6.42c) as

$$\mathcal{A}(Q, b) = \int_{-\infty}^{\infty} dt \left[\frac{1}{2}\dot{Q}^2 - \frac{1}{2}\omega^2 Q^2 + \sqrt{\frac{2\omega}{\hbar}}\; b\,Q + \frac{1}{2}\dot{q}_0^2 - \frac{1}{2}\omega^2 q_0(q_0 - 2\,Q) \right]$$

6.4.3. The Bose Gas

Replacing t with $i\,t$ in eq(5.92) of §5.5, we have

$$\langle \overline{\varphi}'' \mid \boldsymbol{U}(t'', t') \mid \overline{\varphi}' \rangle = \left\langle \overline{\varphi}'' \,\middle|\, \exp\left[-i\,\frac{t'' - t'}{\hbar}\,(\boldsymbol{H} - \mu\,\boldsymbol{N}) \right] \middle|\, \overline{\varphi}' \right\rangle$$

$$= \int [\,d\,\overline{\varphi}(t, \boldsymbol{x})\,d\,\varphi(t, \boldsymbol{x})\,]\,e^{i\,\mathcal{A}(\varphi,\overline{\varphi})/\hbar} \qquad (6.44)$$

where, from eq(5.93),

$$\mathcal{A}(\varphi, \overline{\varphi}) = i\,S\left(\varphi, \overline{\varphi}\right)\Big|_{t \to i\,t}$$

$$= -i\,\hbar\,\varphi(t', \boldsymbol{x}')\,\overline{\varphi}(t', \boldsymbol{x}') - \int_{t'}^{t''} dt \int d\boldsymbol{x}\;\overline{\varphi}(t, \boldsymbol{x}) \left(-i\,\hbar\,\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi(t, \boldsymbol{x}) - \frac{1}{2}\int_{t'}^{t''} dt \int d\boldsymbol{x}\,d\boldsymbol{y}\;\overline{\varphi}(t, \boldsymbol{x})\,\overline{\varphi}(t, \boldsymbol{y})\,V_2(\boldsymbol{x}, \boldsymbol{y})\,\varphi(t, \boldsymbol{y})\,\varphi(t, \boldsymbol{x}) \qquad (6.45)$$

Setting

$$V_1 = 0 \qquad \& \qquad V_2 = G\,\delta(\boldsymbol{x} - \boldsymbol{y})$$

we have

$$\mathcal{A}(\varphi, \overline{\varphi}) = -i\,\hbar\,\varphi(t', \boldsymbol{x}')\,\overline{\varphi}(t', \boldsymbol{x}') - \int_{t'}^{t''} dt \int d\boldsymbol{x} \left\{ \overline{\varphi}(t, \boldsymbol{x}) \left(-i\,\hbar\,\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 - \mu \right) \varphi(t, \boldsymbol{x}) + \frac{1}{2}\,G\,[\,\overline{\varphi}(t, \boldsymbol{x})\,\varphi(t, \boldsymbol{x})\,]^2 \right\} \qquad (6.46)$$

As discussed in §5.5.4, just below the transition temperature, φ is almost classical for small coupling G. The stationary phase (field version of the classical) approximation is valid:

$$\mathcal{L} = \overline{\varphi}(t,\, \boldsymbol{x}) \left(-i\,\hbar\, \frac{\partial}{\partial\, t} - \frac{\hbar^2}{2\,m}\, \nabla_{\boldsymbol{x}}^2 - \mu \right) \varphi(t,\, \boldsymbol{x}) + \frac{1}{2}\, G\, [\, \overline{\varphi}(t,\, \boldsymbol{x})\, \varphi(t,\, \boldsymbol{x})\,]^2$$

$$\doteq \varphi \left(i\,\hbar\, \frac{\partial}{\partial\, t} - \mu \right) \overline{\varphi} + \frac{\hbar^2}{2\,m}\, \nabla_{\boldsymbol{x}}\, \overline{\varphi} \cdot \nabla_{\boldsymbol{x}}\, \varphi + \frac{1}{2}\, G\, (\, \overline{\varphi}\, \varphi\,)^2$$

where $\doteq$ means dynamically equivalent.

Remembering that $\overline{\varphi} \sim z$ is the dynamic variable, we have

$$\frac{\partial\, \mathcal{L}}{\partial\, \dot{\overline{\varphi}}} = i\,\hbar\, \varphi \qquad\qquad \frac{\partial\, \mathcal{L}}{\partial\, \nabla_{\boldsymbol{x}}\, \overline{\varphi}} = \frac{\hbar^2}{2\,m}\, \nabla_{\boldsymbol{x}}\, \varphi$$

$$\frac{\partial\, \mathcal{L}}{\partial\, \overline{\varphi}} = (\, -\mu + G\, \overline{\varphi}\, \varphi\,)\, \varphi$$

The Lagrange eq. for the classical field φ is therefore

$$i\,\hbar\, \frac{\partial}{\partial\, t}\, \varphi + \frac{\hbar^2}{2\,m}\, \nabla_{\boldsymbol{x}}^2\, \varphi - (-\mu + G\, \overline{\varphi}\, \varphi\,)\, \varphi = 0$$

$$i\,\hbar\, \frac{\partial}{\partial\, t}\, \varphi = \left(-\frac{\hbar^2}{2\,m}\, \nabla_{\boldsymbol{x}}^2 - \mu + G\, \rho \right) \varphi \qquad \text{where} \qquad \rho = \overline{\varphi}\, \varphi$$

which is known as the Gross–Pitaevski equation.