6.5. Fermi Gas : Evolution Operator

Eq(5.61) is readily modified to give the real-time version

$$\langle \boldsymbol{\theta}'' \mid U(t'', t') \mid \boldsymbol{\theta}' \rangle = \int_{\boldsymbol{\theta}(t')=\boldsymbol{\theta}'}^{\boldsymbol{\theta}(t'')=\boldsymbol{\theta}''} [\, d\,\boldsymbol{\theta}(t)\, d\,\overline{\boldsymbol{\theta}}(t)\,]\, e^{i\mathcal{A}(\boldsymbol{\theta},\overline{\boldsymbol{\theta}})} \qquad (6.47a)$$

where

$$\mathcal{A}(\boldsymbol{\theta}, \overline{\boldsymbol{\theta}}) = i\,\overline{\boldsymbol{\theta}}(t')\cdot\boldsymbol{\theta}(t') + \int_{t'}^{t''} d\,t \left[\, i\,\overline{\boldsymbol{\theta}}(t)\cdot\dot{\boldsymbol{\theta}}(t) - \frac{h}{\hbar}\,\right] \qquad (6.47b)$$

Note that the free hamiltonian is given by eq(5.52)

$$h_0 = \hbar\,\omega\,\boldsymbol{\theta}\cdot\overline{\boldsymbol{\theta}} = -\hbar\,\omega\,\overline{\boldsymbol{\theta}}\cdot\boldsymbol{\theta} \qquad\qquad \omega > 0 \qquad (6.47c)$$

As in the boson case, $\boldsymbol{\theta}$ & $\overline{\boldsymbol{\theta}}$ can be taken as the creation & annihilation operators, respectively, of a general non-interacting particle state of energy $\hbar\,\omega$. It is therefore customary to set $\omega > 0$.

Given $\omega < 0$, one can always make the transformation

$$\boldsymbol{\eta} = \overline{\boldsymbol{\theta}} \qquad\qquad \overline{\boldsymbol{\eta}} = \boldsymbol{\theta}$$

so that

$$h_0 = \hbar\,\omega\,\overline{\boldsymbol{\eta}}\cdot\boldsymbol{\eta} = -\hbar \mid \omega \mid \overline{\boldsymbol{\eta}}\cdot\boldsymbol{\eta} = \hbar \mid \omega \mid \boldsymbol{\eta}\cdot\overline{\boldsymbol{\eta}}$$

&

$$\mathcal{A}_0(\boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) = i\,\boldsymbol{\eta}(t')\cdot\overline{\boldsymbol{\eta}}(t') + \int_{t'}^{t''} d\,t \left(i\,\boldsymbol{\eta}\cdot\dot{\overline{\boldsymbol{\eta}}} - \frac{h_0}{\hbar} \right)$$

$$= i\,\boldsymbol{\eta}(t'')\cdot\overline{\boldsymbol{\eta}}(t'') + \int_{t'}^{t''} d\,t \left(-i\,\dot{\boldsymbol{\eta}}\cdot\overline{\boldsymbol{\eta}} - \frac{h_0}{\hbar} \right) \qquad \text{[Integration by part.]}$$

$$= -i\,\overline{\boldsymbol{\eta}}(t'')\cdot\boldsymbol{\eta}(t'') + \int_{t'}^{t''} d\,t \left(i\,\overline{\boldsymbol{\eta}}\cdot\dot{\boldsymbol{\eta}} + \mid \omega \mid \overline{\boldsymbol{\eta}}\cdot\boldsymbol{\eta} \right)$$

The Fermi Gas

The counterpart of eqs(6.44-5) is

$$\langle \overline{\varphi}'' \mid \boldsymbol{U}(t'', t') \mid \overline{\varphi}' \rangle = \left\langle \overline{\varphi}'' \,\middle|\, \exp\!\left[-i\,\frac{t''-t'}{\hbar}\,(\boldsymbol{H} - \mu\,\boldsymbol{N}) \right] \middle|\, \overline{\varphi}' \right\rangle$$

$$= \int_{\overline{\varphi}(t')=\overline{\varphi}'}^{\overline{\varphi}(t'')=\overline{\varphi}''} [\, d\,\overline{\varphi}(t, \boldsymbol{x})\, d\,\varphi(t, \boldsymbol{x})\,]\, e^{i\mathcal{A}(\varphi,\overline{\varphi})/\hbar} \qquad (6.48)$$

where (reminder: $\overline{\varphi} \sim \boldsymbol{\theta}$)

$$\mathcal{A}(\varphi, \overline{\varphi}) = -i\,\hbar\,\overline{\varphi}(t', \boldsymbol{x}')\,\varphi(t', \boldsymbol{x}') \qquad (6.49)$$

$$-\int_{t'}^{t''} d\,t \int d\,\boldsymbol{x}\; \overline{\varphi}(t, \boldsymbol{x}) \left(-i\,\hbar\,\frac{\partial}{\partial\,t} - \frac{\hbar^2}{2\,m}\,\nabla_{\boldsymbol{x}}^2 + V_1(\boldsymbol{x}) - \mu \right) \varphi(t, \boldsymbol{x})$$

$$-\frac{1}{2}\int_{t'}^{t''} d\,t \int d\boldsymbol{x}\, d\boldsymbol{y}\; \overline{\varphi}(t, \boldsymbol{x})\, \overline{\varphi}(t, \boldsymbol{y})\, V_2(\boldsymbol{x}, \boldsymbol{y})\, \varphi(t, \boldsymbol{y})\, \varphi(t, \boldsymbol{x})$$

Setting

$$V_1 = 0 \qquad\qquad \& \qquad\qquad V_2 = G\,\delta\,(\boldsymbol{x} - \boldsymbol{y})$$

we have

$$\mathcal{A}(\varphi, \overline{\varphi}) = -i\hbar\,\varphi(t', \boldsymbol{x}')\,\overline{\varphi}(t', \boldsymbol{x}') - \int_{t'}^{t''} dt \int d\boldsymbol{x} \left\{ \overline{\varphi}(t, \boldsymbol{x}) \left(-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla_{\boldsymbol{x}}^2 - \mu \right) \varphi(t, \boldsymbol{x}) + \frac{1}{2} G \left[\overline{\varphi}(t, \boldsymbol{x})\, \varphi(t, \boldsymbol{x}) \right]^2 \right\} \tag{6.50}$$