6.6. Relativistic Quantum Field Theory : The Scalar Field

6.6.1. The Neutral Scalar Field

Let $\phi(t, \boldsymbol{x})$ be a real classical field with $\boldsymbol{x} \in E^{d-1}$ &

$$\mathcal{L}(\phi) = \frac{1}{2c^2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - V(\phi)$$

$$= \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla \phi)^2 - V(\phi) \qquad (6.51)$$

where we've set $c\,t \to t$ (or $c = 1$) so that t has the dimension of length from now on.

V is usually a polynomial function. E.g., the ϕ^4 theory assumes

$$V = \frac{1}{2}\left(\frac{M}{\hbar c}\right)^2 \phi^2 + \frac{1}{4!} g\,\phi^4$$

$$= \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} g\,\phi^4$$

Note that the energy spectrum is bounded below only if V has a minimum. This means either $g > 0$ or $g = 0$ with $m^2 > 0$.

$g = 0$ then gives the free field with M being the mass of the quantum of the associated quantum field. However, $m^2 < 0$ is allowed if $g > 0$.

Properties of the $\mathcal{L}$ in eq(6.51) :

1. It's local both in t & $\boldsymbol{x}$.
2. It's invariant under translation in t & $\boldsymbol{x}$.
3. It's invariant under the pseudo-orthogonal group $O(1, d-1)$.

   [Space-time metric tensor is $\text{diag}\left(1, \overbrace{-1, \ \ldots, -1}^{d-1 \text{ terms}}\right)$.]
4. The associated quantum hamiltonian is hermitian & bounded below.

Quantization & Functional Integrals

Assuming ϕ to be the generalized coordinate, its conjugate momentum is

$$\Pi = \frac{\partial \mathcal{L}}{\partial\, \partial_t \phi} = \partial_t \phi \qquad (6.53)$$

The hamiltonian density is therefore

$$\mathcal{H} = \Pi\, \partial_t \phi - \mathcal{L} \qquad (6.52)$$

$$= \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla \phi)^2 + V(\phi) \qquad (6.56)$$

The total hamiltonian is defined as

$$\boldsymbol{H} = \int d^{d-1} x\, \mathcal{H} \qquad (6.54)$$

Quantization of the field can be achieved by treating π & ϕ as operators satisfying the equal-time commutation relation

$$\left[\hat{\phi}(t, \boldsymbol{x}),\ \hat{\Pi}(t, \boldsymbol{x}')\right] = i\hbar\, \delta^{d-1}(\boldsymbol{x} - \boldsymbol{x}') \qquad (6.55)$$

& is known as the canonical quantization scheme.

Another way to do so is via the path or functional integral:

$$\langle \phi'' \mid U(t'', t') \mid \phi' \rangle = \left\langle \phi'' \mid e^{-i(t''-t')\boldsymbol{H}/\hbar} \mid \phi' \right\rangle$$

$$= \int_{\phi(t',\boldsymbol{x})=\phi'(\boldsymbol{x})}^{\phi(t'',\boldsymbol{x})=\phi''(\boldsymbol{x})} [d\,\phi(t,\boldsymbol{x})]\, e^{i\mathcal{A}(\phi)/\hbar} \qquad (6.57)$$

where

$$\mathcal{A}(\phi) = \int_{t'}^{t''} dt \int d^{d-1}x\, \mathcal{L}$$

$$= \int_{t'}^{t''} dt \int d\boldsymbol{x} \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right] \qquad (6.58)$$

6.6.2. Free Field Theory & the Holomorphic Formalism

Consider the free field action

$$\mathcal{A}_0(\phi) = \int_{t'}^{t''} dt \int d\boldsymbol{x}\, \mathcal{L}_0$$

$$= \int_{t'}^{t''} dt \int d\boldsymbol{x}\, \frac{1}{2}\left[(\partial_t\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right] \qquad (6.59)$$

$$= \int_{t'}^{t''} dt \int d\boldsymbol{x}\, (\Pi\,\partial_t\phi - \mathcal{H}_0)$$

$$= \int_{t'}^{t''} dt \int d\boldsymbol{x} \left\{ \Pi\,\partial_t\phi - \frac{1}{2}\left[\Pi^2 + (\nabla\phi)^2 + m^2\phi^2\right] \right\} \qquad (6.60)$$

$$= \mathcal{A}_0(\Pi, \phi)$$

Using

$$\frac{\partial\mathcal{L}_0}{\partial\,\partial_t\phi} = \partial_t\phi \qquad \frac{\partial\mathcal{L}_0}{\partial\,\nabla\phi} = -\nabla\phi \qquad \frac{\partial\mathcal{L}_0}{\partial\phi} = -m^2\phi$$

we get the Lagrange eq.

$$\partial_t^2\phi - \nabla^2\phi + m^2\phi = 0 \qquad (6.60a)$$

which is known as the Klein-Gordon (K-G) eq.

Solutions to eq(6.60a) are plane waves:

$$e^{\pm(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega_{\boldsymbol{k}} t)} \quad \text{with} \quad \omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 + m^2} \qquad (6.60b)$$

Note that $c = 1$ implicitly.

Introducing the 4-vectors [see e.g., I.D.Lawrie, “A Unified Grand Tour of Theoretical Physics”, §7.1 or M.Kaku, “Quantum Field Theory”, §3.2.]

$$k^\mu = (k_0, \boldsymbol{k}) = (\omega_{\boldsymbol{k}}, \boldsymbol{k}) \qquad k_\mu = (k_0, -\boldsymbol{k}) = (\omega_{\boldsymbol{k}}, -\boldsymbol{k})$$

$$x^\mu = (t, \boldsymbol{x}) \qquad x_\mu = (t, -\boldsymbol{x})$$

we have

$$k^2 = k^\mu k_\mu = k_0^2 - \boldsymbol{k}^2 = m^2 \qquad (6.62)$$

$$k\cdot x = k_\mu x^\mu = k_0 t - \boldsymbol{k}\cdot\boldsymbol{x} = \omega_{\boldsymbol{k}} t - \boldsymbol{k}\cdot\boldsymbol{x}$$

Note that formally,

$$p^\mu = \hbar k^\mu = (p_0, \boldsymbol{p}) = \left(\frac{E}{c}, \boldsymbol{p}\right) = \left(\frac{\hbar\omega_{\boldsymbol{k}}}{c}, \hbar\boldsymbol{k}\right)$$

However, we’ll be using the natural units with $\hbar = c = 1$.

As a constraint, eq(6.62) can be written as

$$\delta(k^2 - m^2) = \frac{1}{2\,\omega_{\boldsymbol{k}}}\left\{ \delta(k_0 - \omega_{\boldsymbol{k}}) + \delta(k_0 + \omega_{\boldsymbol{k}}) \right\} \qquad (6.62a)$$

Let

$$\delta_+\left(k^2-m^2\right)=\delta\left(k^2-m^2\right)\theta(k_0) \qquad (6.62b)$$

$$=\frac{1}{2\,\omega_{\boldsymbol{k}}}\,\delta(\,k_0-\omega_{\boldsymbol{k}}\,)$$

then

$$\int d\,k\,\delta_+\left(k^2-m^2\right)f(\boldsymbol{k})=\int\frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\,f(\boldsymbol{k}) \qquad (6.62c)$$

The covariant plane wave expansion of a general solution ϕ is therefore [see §“Notations” below]

$$\phi(t,\,\boldsymbol{x})=\int\frac{d\,k}{(2\,\pi)^{d/2}}\,\sqrt{4\,\pi\,\omega_{\boldsymbol{k}}}\;\delta\left(k^2-m^2\right)e^{-i\,k\cdot x}\,\varphi(k)$$

$$=\int\frac{d^{\,d-1}\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\left\{e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(\omega_{\boldsymbol{k}},\,\boldsymbol{k})+e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}+i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(-\omega_{\boldsymbol{k}},\,\boldsymbol{k})\right\}$$

Using $\omega_{-\boldsymbol{k}}=\omega_{\boldsymbol{k}}$, we have

$$\phi(t,\,\boldsymbol{x})=\int\frac{d\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\left\{e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(\omega_{\boldsymbol{k}},\,\boldsymbol{k})+e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}+i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(-\omega_{\boldsymbol{k}},\,-\boldsymbol{k})\right\}$$

Since ϕ is real, we have

$$\varphi(-\omega_{\boldsymbol{k}},\,-\boldsymbol{k})=\overline{\varphi}(\omega_{\boldsymbol{k}},\,\boldsymbol{k}) \qquad (6.62d)$$

so that

$$\phi(t,\,\boldsymbol{x})=\int\frac{d\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\left[e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(\boldsymbol{k})+e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}+i\,\omega_{\boldsymbol{k}}\,t}\,\overline{\varphi}(\boldsymbol{k})\right] \qquad (6.61a)$$

→ $$\Pi(t,\,\boldsymbol{x})=\partial_t\,\phi$$

$$=i\int\frac{d\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\,\omega_{\boldsymbol{k}}\left[-e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(\boldsymbol{k})+e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}+i\,\omega_{\boldsymbol{k}}\,t}\,\overline{\varphi}(\boldsymbol{k})\right] \qquad (6.61b)$$

$$\nabla\phi=i\int\frac{d\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\,\boldsymbol{k}\left[e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}-i\,\omega_{\boldsymbol{k}}\,t}\,\varphi(\boldsymbol{k})-e^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}+i\,\omega_{\boldsymbol{k}}\,t}\,\overline{\varphi}(\boldsymbol{k})\right]$$

$\mathcal{H}_0$ can be evaluated as follows:

$$\int d\,\boldsymbol{x}\,(\partial_t\,\phi)^2=-\int d\,\boldsymbol{x}\int\frac{d\,\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}}}}\int\frac{d\,\boldsymbol{k}'}{\sqrt{(2\,\pi)^{d-1}\,2\,\omega_{\boldsymbol{k}'}}}\,\omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{k}'}$$

$$\times\left[e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}-i\,(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\,t}\,\varphi(\boldsymbol{k})\,\varphi(\boldsymbol{k}')-e^{i\,(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}\,\varphi(\boldsymbol{k})\,\overline{\varphi}(\boldsymbol{k}')\right.$$

$$\left.-\,e^{i\,(-\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}+i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}\,\overline{\varphi}(\boldsymbol{k})\,\varphi(\boldsymbol{k}')+e^{-i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x}+i\,(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\,t}\,\overline{\varphi}(\boldsymbol{k})\,\overline{\varphi}(\boldsymbol{k}')\right]$$

$$=-\int\frac{d\,\boldsymbol{k}}{\sqrt{2\,\omega_{\boldsymbol{k}}}}\int\frac{d\,\boldsymbol{k}'}{\sqrt{2\,\omega_{\boldsymbol{k}'}}}\,\omega_{\boldsymbol{k}}\,\omega_{\boldsymbol{k}'}$$

$$\times\left[\delta(\boldsymbol{k}+\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\,t}\,\varphi(\boldsymbol{k})\,\varphi(\boldsymbol{k}')-\delta\,(\boldsymbol{k}-\boldsymbol{k}')\,e^{-i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}\,\varphi(\boldsymbol{k})\,\overline{\varphi}(\boldsymbol{k}')\right.$$

$$\left.-\,\delta\,(\boldsymbol{k}-\boldsymbol{k}')\,e^{\,i\,(\omega_{\boldsymbol{k}}-\omega_{\boldsymbol{k}'})\,t}\,\overline{\varphi}(\boldsymbol{k})\,\varphi(\boldsymbol{k}')+\delta\,(\boldsymbol{k}+\boldsymbol{k}')\,e^{\,i\,(\omega_{\boldsymbol{k}}+\omega_{\boldsymbol{k}'})\,t}\,\overline{\varphi}(\boldsymbol{k})\,\overline{\varphi}(\boldsymbol{k}')\right]$$

$$=-\int\frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\,\omega_{\boldsymbol{k}}^2\left[\varphi(\boldsymbol{k})\,\varphi(-\boldsymbol{k})\,e^{-2\,i\,\omega_{\boldsymbol{k}}\,t}-\varphi(\boldsymbol{k})\,\overline{\varphi}(\boldsymbol{k})\right.$$

$$\left.-\,\overline{\varphi}(\boldsymbol{k})\,\varphi(\boldsymbol{k})+e^{2\,i\,\omega_{\boldsymbol{k}}\,t}\,\overline{\varphi}(\boldsymbol{k})\,\overline{\varphi}(-\boldsymbol{k})\right]$$

$$= -\int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \omega_{\boldsymbol{k}}^2\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}) \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} - 2 \right) \qquad \text{[Eq(6.62d) used.]}$$

$$\int d\boldsymbol{x}\, (\nabla\phi)^2 = -\int d\boldsymbol{x} \int \frac{d\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\, 2\,\omega_{\boldsymbol{k}}}} \int \frac{d\boldsymbol{k}'}{\sqrt{(2\,\pi)^{d-1}\, 2\,\omega_{\boldsymbol{k}'}}}\, \boldsymbol{k}\cdot\boldsymbol{k}'$$
$$\times \Big[e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} - i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \varphi(\boldsymbol{k}') - e^{i\,(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x} - i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}')$$
$$- e^{i\,(-\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} + i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + e^{-i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} + i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}') \Big]$$

$$= -\int \frac{d\boldsymbol{k}}{\sqrt{2\,\omega_{\boldsymbol{k}}}} \int \frac{d\boldsymbol{k}'}{\sqrt{2\,\omega_{\boldsymbol{k}'}}}\, \boldsymbol{k}\cdot\boldsymbol{k}'$$
$$\times \Big[\delta(\boldsymbol{k}+\boldsymbol{k}')\, e^{-i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \varphi(\boldsymbol{k}') - \delta(\boldsymbol{k}-\boldsymbol{k}')\, e^{-i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}')$$
$$- \delta(\boldsymbol{k}-\boldsymbol{k}')\, e^{i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + \delta(\boldsymbol{k}+\boldsymbol{k}')\, e^{i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}') \Big]$$

$$= -\int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \boldsymbol{k}^2 \Big[-\varphi(\boldsymbol{k})\, \varphi(-\boldsymbol{k})\, e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} - \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k})$$
$$- \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}) - e^{2\,i\,\omega_{\boldsymbol{k}}\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(-\boldsymbol{k}) \Big]$$

$$= \int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \boldsymbol{k}^2\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}) \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} + 2 \right) \qquad \text{[Eq(6.62d) used.]}$$

$$\int d\boldsymbol{x}\, \phi^2 = \int d\boldsymbol{x} \int \frac{d\boldsymbol{k}}{\sqrt{(2\,\pi)^{d-1}\, 2\,\omega_{\boldsymbol{k}}}} \int \frac{d\boldsymbol{k}'}{\sqrt{(2\,\pi)^{d-1}\, 2\,\omega_{\boldsymbol{k}'}}}$$
$$\times \Big[e^{i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} - i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + e^{i\,(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x} - i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}')$$
$$+ e^{i\,(-\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} + i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + e^{-i\,(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{x} + i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}') \Big]$$

$$= \int \frac{d\boldsymbol{k}}{\sqrt{2\,\omega_{\boldsymbol{k}}}} \int \frac{d\boldsymbol{k}'}{\sqrt{2\,\omega_{\boldsymbol{k}'}}}$$
$$\times \Big[\delta(\boldsymbol{k}+\boldsymbol{k}')\, e^{-i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + \delta(\boldsymbol{k}-\boldsymbol{k}')\, e^{-i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}')$$
$$+ \delta(\boldsymbol{k}-\boldsymbol{k}')\, e^{i\,(\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}') + \delta(\boldsymbol{k}+\boldsymbol{k}')\, e^{i\,(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'})\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k}') \Big]$$

$$= \int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} \Big[\varphi(\boldsymbol{k})\, \varphi(-\boldsymbol{k})\, e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + \varphi(\boldsymbol{k})\, \overline{\varphi}(\boldsymbol{k})$$
$$+ \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}) + e^{2\,i\,\omega_{\boldsymbol{k}}\,t}\, \overline{\varphi}(\boldsymbol{k})\, \overline{\varphi}(-\boldsymbol{k}) \Big]$$

$$= \int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k}) \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} + 2 \right) \qquad \text{[Eq(6.62d) used.]}$$

Hence,

$$\mathcal{A}_0(\phi) = \frac{1}{2} \int_{t'}^{t''} d\,t \int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k})$$
$$\times \Big[-\omega_{\boldsymbol{k}}^2 \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} - 2 \right) - \boldsymbol{k}^2 \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} + 2 \right)$$
$$- m^2 \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} + 2 \right) \Big]$$
$$= -\int_{t'}^{t''} d\,t \int \frac{d\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k})\, \omega_{\boldsymbol{k}}^2 \left(e^{-2\,i\,\omega_{\boldsymbol{k}}\,t} + e^{2\,i\,\omega_{\boldsymbol{k}}\,t} \right)$$
$$= -\int_{t'}^{t''} d\,t \int d\boldsymbol{k}\, \overline{\varphi}(\boldsymbol{k})\, \varphi(\boldsymbol{k})\, \omega_{\boldsymbol{k}} \cos(2\,\omega_{\boldsymbol{k}}\,t) \qquad (6.63a)$$

Note that $\mathcal{A}_0$ in eq(6.63a) is no longer explicitly local in space.

To compare with Zinn-Justin's eq(6.63), we set

$$\varphi(\boldsymbol{k},t)=\varphi(\boldsymbol{k})\,e^{-i\,\omega_k t} \qquad \overline{\varphi}(\boldsymbol{k},t)=\overline{\varphi}(\boldsymbol{k})\,e^{i\,\omega_k t} \qquad (6.63b)$$

as required to get eq(6.61b) from eq(6.61a).

$\rightarrow \quad i\,\overline{\varphi}(\boldsymbol{k},t)\,\partial_t\,\varphi(\boldsymbol{k},t)=\omega_{\boldsymbol{k}}\,\overline{\varphi}(\boldsymbol{k})\,\varphi\,(\boldsymbol{k})$

Hence, eqs(6.63a & 6.63) are not equivalent.

Using the above result, we get

$$\begin{aligned}
\boldsymbol{H}_0 &= \int d\,\boldsymbol{x}\,\frac{1}{2}\left[(\partial_t\,\phi)^2+(\nabla\,\phi)^2+m^2\,\phi^2\right] \\
&= \frac{1}{2}\int\frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\,\overline{\varphi}(\boldsymbol{k})\,\varphi(\boldsymbol{k}) \\
&\qquad \times\left[-\omega_{\boldsymbol{k}}^2\left(e^{-2\,i\,\omega_k t}+e^{2\,i\,\omega_k t}-2\right)+\boldsymbol{k}^2\left(e^{-2\,i\,\omega_k t}+e^{2\,i\,\omega_k t}+2\right)\right. \\
&\qquad\qquad \left.+\,m^2\left(e^{-2\,i\,\omega_k t}+e^{2\,i\,\omega_k t}+2\right)\right] \\
&= \int d\,\boldsymbol{k}\,\overline{\varphi}(\boldsymbol{k})\,\varphi(\boldsymbol{k})\,\omega_{\boldsymbol{k}} \qquad (6.63c)
\end{aligned}$$

as expected.

Notations

Lawrie's choice:

$$\mathcal{L}_0\,(\phi)=(\partial_t\,\phi)^2-(\nabla\,\phi)^2-m^2\,\phi^2$$

$$\phi(t,\boldsymbol{x})=\int\frac{d\,k}{(2\,\pi)^d}\,2\,\pi\,\delta\!\left(k^2-m^2\right)e^{-i\,k\cdot x}\,\varphi(k)$$

Kaku & our choice:

$$\mathcal{L}_0\,(\phi)=\frac{1}{2}\left[(\partial_t\,\phi)^2-(\nabla\,\phi)^2-m^2\,\phi^2\right]$$

$$\phi(t,\boldsymbol{x})=\int\frac{d\,k}{(2\,\pi)^{d/2}}\sqrt{4\,\pi\,\omega_{\boldsymbol{k}}}\;\delta\!\left(k^2-m^2\right)e^{-i\,k\cdot x}\,\varphi(k)$$

Both choices will lead to the correct form of $\boldsymbol{H}_0$.

Zinn-Justin's choice corresponds to a mixture of the two choices & the resultant $\boldsymbol{H}_0$ is off by a factor of 2.

Reminder:

Our convention is that

$$f(\boldsymbol{x})=\int\frac{d^d\,k}{(2\,\pi)^d}\,f(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \text{for the fourier transform of a function } f.$$

$$\phi(\boldsymbol{x})=\int\frac{d^d\,k}{(2\,\pi)^{d/2}}\,\phi(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \text{for a plane wave expansion of } \phi.$$

$$=\int\frac{d^d\,k}{\sqrt{(2\,\pi)^d\,2\,\omega_{\boldsymbol{k}}}}\,\phi(\boldsymbol{k})\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} \qquad \text{covariant normalization}$$

Functional Integral

As in §6.4, we can, by treating $\overline{\varphi}$ as z, construct the holomorphic path (or functional) integral representation of the evolution operator:

$$\langle \overline{\varphi}'' \mid U_0(t'', t') \mid \overline{\varphi}' \rangle = \int \left[\frac{d\,\varphi(t, \boldsymbol{k})\, d\,\overline{\varphi}(t, \boldsymbol{k})}{2\,\omega_{\boldsymbol{k}}} \right] e^{i \mathcal{A}_0} \tag{6.63d}$$

where $\mathcal{A}_0$ is given in eq(6.63a).

Fock's Space

The plane wave expansion eq(6.61a) implies the completeness relations

$$\int d\,k \mid k \rangle \langle k \mid = I$$

$$\int d\,k\, \delta(k^2 - m^2) \mid k \rangle \langle k \mid = I_S$$

$$= \int \frac{d^{\,d-1}\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} (\mid k_+ \rangle \langle k_+ \mid + \mid k_- \rangle \langle k_- \mid) \tag{6.64a}$$

where I_S is the identity operator in the solution space of the K-G eq(6.60a) &

$$\mid \boldsymbol{k}_\pm \rangle \equiv \mid \pm \omega_{\boldsymbol{k}}, \boldsymbol{k} \rangle$$

Eq(6.61a) itself is then obtained from

$$\phi(t, \boldsymbol{x}) = \langle x \mid \phi \rangle$$

$$= \int \frac{d^{\,d-1}\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} (\langle x \mid k_+ \rangle \langle k_+ \mid \phi \rangle + \langle x \mid k_- \rangle \langle k_- \mid \phi \rangle)$$

where

$$\langle x \mid k \rangle = \sqrt{\frac{2 \mid k_0 \mid}{(2\,\pi)^{d-1}}}\, e^{-i\,k \cdot x} \qquad \langle k_\pm \mid \phi \rangle = \varphi\,(k_\pm)$$

$$\langle x \mid k_\pm \rangle = \sqrt{\frac{2\,\omega_{\boldsymbol{k}}}{(2\,\pi)^{d-1}}}\, e^{i\,\boldsymbol{k} \cdot \boldsymbol{x} \mp i\,\omega_{\boldsymbol{k}}\,t}$$

If we define

$$\langle \boldsymbol{x} \mid \boldsymbol{k} \rangle = \sqrt{\frac{2\,\omega_{\boldsymbol{k}}}{(2\,\pi)^{d-1}}}\, e^{i\,\boldsymbol{k} \cdot \boldsymbol{x}}$$

the completeness of $\{ \mid \boldsymbol{k} \rangle \}$ must be modified to

$$\int \frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} \mid \boldsymbol{k} \rangle \langle \boldsymbol{k} \mid = I \tag{6.64b}$$

in order to keep up the delta-normalization of $\{ \mid \boldsymbol{x} \rangle \}$:

$$\int \frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} \langle \boldsymbol{x} \mid \boldsymbol{k} \rangle \langle \boldsymbol{k} \mid \boldsymbol{x}' \rangle = \langle \boldsymbol{x} \mid \boldsymbol{x}' \rangle = \delta(\boldsymbol{x} - \boldsymbol{x}')$$

$$= \int \frac{d\,\boldsymbol{k}}{(2\,\pi)^{d-1}}\, e^{i\,\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}')}$$

Note that the plane waves are now normalized as

$$\langle \boldsymbol{k} \mid \boldsymbol{k}' \rangle = \int d\,\boldsymbol{x} \langle \boldsymbol{k} \mid \boldsymbol{x} \rangle \langle \boldsymbol{x} \mid \boldsymbol{k}' \rangle$$

$$= \frac{2\,\omega_{\boldsymbol{k}}}{(2\,\pi)^{d-1}} \int d\,\boldsymbol{x}\, e^{i\,(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{x}}$$

$$= 2\,\omega_{\boldsymbol{k}}\, \delta(\boldsymbol{k} - \boldsymbol{k}') \tag{6.64c}$$

Eqs(6.64b-c) are sometimes called the covariant normalization.

Let $\mid \psi \rangle$ be any 1-particle state. Using eq(6.64b), we have

$$\langle \psi_1 \mid \psi_2 \rangle = \int \frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}} \langle \psi_1 \mid \boldsymbol{k} \rangle \langle \boldsymbol{k} \mid \psi_2 \rangle$$

$$= \int \frac{d\,\boldsymbol{k}}{2\,\omega_{\boldsymbol{k}}}\, \overline{\psi}_1(\boldsymbol{k})\, \psi_2(\boldsymbol{k}) \qquad (6.64)$$

Continuing the identification of $\overline{\varphi}$ with z, we put eq(6.63c) into eqs(6.38a-b) of §6.4.1 to get

$$U_0(\overline{\varphi}'', \overline{\varphi}'; t'', t') = \langle \overline{\varphi}'' \mid U_0(t'', t') \mid \overline{\varphi}' \rangle$$

$$= \exp\left(\int d\,\boldsymbol{k}\, e^{-i\,\omega_{\boldsymbol{k}}\,(t''-t')}\, \overline{\varphi}''\, \varphi' \right) \qquad (6.65a)$$

Let $\varphi(\boldsymbol{k})$ be a complex field. The generating functional for the bosonic Fock space is defined as [see eq(5.86) of §5.5]

$$\boldsymbol{\Psi}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n) \prod_{i=1}^{n} \frac{d\,\boldsymbol{k}_i}{2\,\omega_{\boldsymbol{k}_i}}\, \varphi(\boldsymbol{k}_i) \qquad (6.65)$$

where $\psi(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_n)$ is totally symmetric in its arguments.

The $\boldsymbol{k}$-representation of eq(5.90s) of §5.5.2, is

$$\int [\,d\,\varphi(\boldsymbol{k})\, d\,\overline{\varphi}(\boldsymbol{k})\,] \exp\left(-\int d\,\boldsymbol{k}\ \overline{\varphi}\,\varphi\right) \mid \overline{\varphi} \rangle \langle \overline{\varphi} \mid = I \qquad (6.66a)$$

so that the inner product is defined as

$$\langle \boldsymbol{\Psi}_2 \mid \boldsymbol{\Psi}_1 \rangle = \int [\,d\,\varphi(\boldsymbol{k})\, d\,\overline{\varphi}(\boldsymbol{k})\,] \exp\left(-\int d\,\boldsymbol{k}\ \overline{\varphi}\,\varphi\right) \langle \boldsymbol{\Psi}_2 \mid \overline{\varphi} \rangle \langle \overline{\varphi} \mid \boldsymbol{\Psi}_1 \rangle$$

$$= \int [\,d\,\varphi(\boldsymbol{k})\, d\,\overline{\varphi}(\boldsymbol{k})\,] \exp\left(-\int d\,\boldsymbol{k}\ \overline{\varphi}\,\varphi\right) \overline{\boldsymbol{\Psi}_2(\overline{\varphi})}\, \boldsymbol{\Psi}_1(\overline{\varphi}) \qquad (6.66)$$

$$= \int [\,d\,\varphi(\boldsymbol{k})\, d\,\overline{\varphi}(\boldsymbol{k})\,] \exp\left(-\int d\,\boldsymbol{k}\ \overline{\varphi}\,\varphi\right) \overline{\boldsymbol{\Psi}_1(\varphi)}\, \boldsymbol{\Psi}_2(\varphi)$$

which is the complex conjugate to Zinn-Justin’s version.

Operators

As in §5.5.2, we set [see eqs(5.87a & b)]

$$\hat{\boldsymbol{O}} = \int d\,\boldsymbol{k}\, \overline{\varphi}(\boldsymbol{k})\, \boldsymbol{O}(\boldsymbol{k}) \frac{\delta}{\delta\, \overline{\varphi}(\boldsymbol{k})} \qquad (6.67a)$$

For example,

$$\hat{\boldsymbol{H}}_0 = \int d\,\boldsymbol{k}\, \overline{\varphi}(\boldsymbol{k})\, \omega_{\boldsymbol{k}} \frac{\delta}{\delta\, \overline{\varphi}(\boldsymbol{k})} + E_0 \qquad (6.67b)$$

where we’ve added by hand the vacuum energy E_0 &

$$\hat{\boldsymbol{N}} = \int d\,\boldsymbol{k}\, \overline{\varphi}(\boldsymbol{k}) \frac{\delta}{\delta\, \overline{\varphi}(\boldsymbol{k})} \qquad (6.67c)$$

The $\boldsymbol{k}$-representation of the kernel eq(5.90t) is

$$\langle \overline{\varphi} \mid \overline{\varphi}' \rangle = \exp\left(\int d\,\boldsymbol{k}\, \overline{\varphi}\, \varphi' \right) \qquad (6.67)$$

so that [see eqs(5.90-1)]

$$\langle \overline{\varphi} \mid \boldsymbol{H}_0 \mid \overline{\varphi}' \rangle = \hat{\boldsymbol{H}}_0 \langle \overline{\varphi} \mid \overline{\varphi}' \rangle$$

$$= \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \left(\int d\,\boldsymbol{k}\, \omega_{\boldsymbol{k}}\, \overline{\varphi}\, \varphi' + E_0 \right) \qquad (6.67d)$$

$$\langle \overline{\varphi} \mid \boldsymbol{N} \mid \overline{\varphi}' \rangle = \langle \overline{\varphi} \mid \overline{\varphi}' \rangle \int d\,\boldsymbol{k}\, \overline{\varphi}\, \varphi' \qquad (6.67e)$$

The Vacuum Energy

The vacuum energy is the energy of the state with no particles.

$$\boldsymbol{H}_0 \mid 0 \rangle = E_0 \mid 0 \rangle$$

Treating the field as a set of harmonic oscillators, we have

$$E_0 = \frac{1}{2} \int d\boldsymbol{k}\, \omega_{\boldsymbol{k}} = \frac{1}{2} \int d\boldsymbol{k} \sqrt{m^2 + \boldsymbol{k}^2}$$

$$= \Omega_{d-1} \int_0^\infty dk \frac{k^{d-2}}{\sqrt{m^2 + k^2}}$$

$$= \Omega_{d-1} \begin{cases} \ln\left(k + \sqrt{k^2 + m^2}\right)\Big|_0^\infty & \text{for } d = 2 \\ \sqrt{k^2 + m^2}\,\Big|_0^\infty & \text{for } d = 3 \\ \frac{1}{2} k \sqrt{k^2 + m^2} - \frac{1}{2} m^2 \ln\left(k + \sqrt{k^2 + m^2}\right)\Big|_0^\infty & \text{for } d = 4 \end{cases}$$

$$= \infty$$

E_0 can be made finite by introducing a cut-off $k_{max} = \Lambda$.
In problems not involving gravitational fields, E_0 can be dropped by shifting the origin of the energy scale.

Two-Point Function

A useful 2-point (or green) function is

$$\langle 0 \mid T[\varphi(t, \boldsymbol{k})\, \varphi(0, \boldsymbol{k}')] \mid 0 \rangle$$

where T is the time-ordered operator defined as

$$T[f(t)\, g(t')] = \begin{cases} f g & \text{for } t > t' \\ g f & \text{for } t < t' \end{cases}$$

$$= \theta(t - t') f g + \theta(t' - t) g f$$

&

$$\varphi(t, \boldsymbol{k}) = e^{i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k})\, e^{-i \boldsymbol{H}_0 t}$$

is the Heisenberg interaction picture version of the free field $\varphi(\boldsymbol{k})$.

Hence,

$$T[\varphi(t, \boldsymbol{k})\, \varphi(0, \boldsymbol{k}')] = \begin{cases} \varphi(t, \boldsymbol{k})\, \varphi(0, \boldsymbol{k}') & \text{for } t > 0 \\ \varphi(0, \boldsymbol{k}')\, \varphi(t, \boldsymbol{k}) & \text{for } t < 0 \end{cases}$$

$$= \begin{cases} e^{i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k})\, e^{-i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k}') & \text{for } t > 0 \\ \varphi(\boldsymbol{k}')\, e^{i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k})\, e^{-i \boldsymbol{H}_0 t} & \text{for } t < 0 \end{cases}$$

Setting $E_0 = 0$, we have

$$\langle 0 \mid T[\varphi(t, \boldsymbol{k})\, \varphi(0, \boldsymbol{k}')] \mid 0 \rangle = \begin{cases} \langle 0 \mid \varphi(\boldsymbol{k})\, e^{-i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k}') \mid 0 \rangle & \text{for } t > 0 \\ \langle 0 \mid \varphi(\boldsymbol{k}')\, e^{i \boldsymbol{H}_0 t} \varphi(\boldsymbol{k}) \mid 0 \rangle & \text{for } t < 0 \end{cases}$$

$$= \langle 0 \mid \varphi(\boldsymbol{k})\, e^{-i \boldsymbol{H}_0 |t|} \varphi(\boldsymbol{k}') \mid 0 \rangle \qquad (6.69)$$

The analog of eq(6.41) is

$$\langle \overline{\varphi}'' \mid \mathcal{S} \mid \overline{\varphi}' \rangle = \exp\left\{ i \int_{-\infty}^{\infty} d\,t\, b(\mathbf{k}', t)\, \varphi(\mathbf{k}')\, e^{-i\,\omega_{\mathbf{k}'}\,t} - \frac{1}{2} \int \frac{d\,\mathbf{k}}{2\,\omega_{\mathbf{k}}} \int_{-\infty}^{\infty} d\,t \int_{-\infty}^{\infty} d\,u\, \overline{b}(\mathbf{k}, t)\, e^{-i\,\omega_{\mathbf{k}}\,|\,t-u\,|}\, b(\mathbf{k}, u) \right)$$

$$\therefore \qquad \langle 0 \mid T[\,\varphi(t, \mathbf{k})\, \overline{\varphi}(0, \mathbf{k}')\,] \mid 0 \rangle = \frac{\delta}{\delta\, \overline{b}(\mathbf{k}, t)} \frac{\delta}{\delta\, b(\mathbf{k}', 0)} \langle 0 \mid \mathcal{S} \mid 0 \rangle \Big|_{b=0} = \frac{1}{2\,\omega_{\mathbf{k}}}\, \delta(\mathbf{k} - \mathbf{k}')\, e^{-i\,\omega_{\mathbf{k}}\,|\,t\,|} \qquad (6.69a)$$

The fourier transform of eq(6.69a) is

$$\int d\,t\, e^{i\,k_0\,t} \langle 0 \mid T[\,\varphi(t, \mathbf{k})\, \overline{\varphi}(0, \mathbf{k}')\,] \mid 0 \rangle = \frac{1}{2\,\omega_{\mathbf{k}}}\, \delta(\mathbf{k} - \mathbf{k}') \int d\,t\, e^{i\,k_0\,t - i\,\omega_{\mathbf{k}}\,|\,t\,|}$$

$$= \frac{1}{2\,\omega_{\mathbf{k}}}\, \delta(\mathbf{k} - \mathbf{k}') \left(\int_{-\infty}^{0} d\,t\, e^{i\,(k_0 + \omega_{\mathbf{k}})\,t} + \int_{0}^{\infty} d\,t\, e^{i\,(k_0 - \omega_{\mathbf{k}})\,t} \right)$$

$$= \frac{1}{2\,\omega_{\mathbf{k}}}\, \delta(\mathbf{k} - \mathbf{k}') \left(\frac{1}{i\,(k_0 + \omega_{\mathbf{k}} - i\,\varepsilon)} - \frac{1}{i\,(k_0 - \omega_{\mathbf{k}} + i\,\varepsilon\,)} \right)$$

$$= \delta(\mathbf{k} - \mathbf{k}')\, \frac{i}{k_0^2 - \omega_{\mathbf{k}}^2 + i\,\varepsilon} \qquad (6.70)$$

where we’ve used

$$\omega_{\mathbf{k}}\, \varepsilon = \varepsilon \qquad \text{as} \qquad \varepsilon \to 0$$

Using

$$\frac{1}{x + i\,\varepsilon} = P\,\frac{1}{x} - i\,\pi\,\delta(x)$$

where P means taking the principal value, we have

$$\frac{i}{k_0^2 - \omega_{\mathbf{k}}^2 + i\,\varepsilon} = i\,P\,\frac{1}{k_0^2 - \omega_{\mathbf{k}}^2} + \pi\,\delta\left(k_0^2 - \omega_{\mathbf{k}}^2\right) \qquad (6.70a)$$