1.5. Coherent States And Von Neumann Lattice

Coherent state $|\, v \rangle$ is an eigenstate of a :

$$a\,|\, v\rangle = v\;|\, v\rangle \qquad v \in \mathbb{C}$$

$$\langle v\,|\, a^{+} = \langle v\,|\, v^{*}$$

Let $\quad a = v + \eta$

$\rightarrow \quad (v+\eta)\,|\, v\rangle = v\;|\, v\rangle$

$\therefore \quad \eta\,|\, v\rangle = 0 \quad$ and $\quad \langle v\,|\,\eta^{+} = 0$

$$[a, a^{+}] = 1 \;= [v+\eta, v^{*}+\eta^{+}] \;= [\eta, \eta^{+}]$$

Bosons:

$$H = \frac{1}{2M}p^{2} + \frac{1}{2}M\,\omega^{2}\,q^{2} = \hbar\,\omega\left(a^{+}\,a + \frac{1}{2}\right)$$

$$a = \frac{1}{\sqrt{2\,M\,\hbar\,\omega}}(M\,\omega\,q + i\,p)$$

$$a^{+} = \frac{1}{\sqrt{2\,M\,\hbar\,\omega}}(M\,\omega\,q - i\,p)$$

$\rightarrow$

$$q = \sqrt{\frac{\hbar}{2\,M\,\omega}}\,\left(a + a^{+}\right)$$

$$p = -i\sqrt{\frac{M\,\hbar\,\omega}{2}}\,\left(a - a^{+}\right)$$

$\therefore$

$$q = \sqrt{\frac{\hbar}{2\,M\,\omega}}\,\left(v + \eta + v^{*} + \eta^{+}\right)$$

$$\langle q\rangle = \langle v\,|\, q\,|\, v\rangle \;= \sqrt{\frac{\hbar}{2\,M\,\omega}}\,(v + v^{*})$$

$$\Delta q = q - \langle q\rangle \;= \sqrt{\frac{\hbar}{2\,M\,\omega}}\,\left(\eta + \eta^{+}\right)$$

$$\left\langle (\Delta q)^{2}\right\rangle = \frac{\hbar}{2\,M\,\omega}\left\langle \eta^{2} + \eta^{+2} + \eta\,\eta^{+} + \eta^{+}\,\eta\right\rangle$$

$$= \frac{\hbar}{2\,M\,\omega}\left\langle \eta\,\eta^{+}\right\rangle \;= \frac{\hbar}{2\,M\,\omega}\left\langle 1 + \eta^{+}\,\eta\right\rangle$$

$$= \frac{\hbar}{2\,M\,\omega}$$

$$p = -i\sqrt{\frac{M\,\hbar\,\omega}{2}}\,\left(v + \eta - v^{*} - \eta^{+}\right)$$

$$\langle p\rangle = \langle v\,|\, p\,|\, v\rangle \;= -i\sqrt{\frac{M\,\hbar\,\omega}{2}}\,(v - v^{*})$$

$$\Delta p = p - \langle p \rangle = -i \sqrt{\frac{M \hbar \omega}{2}} \left(\eta - \eta^{+}\right)$$

$$\left\langle (\Delta p)^2 \right\rangle = -\frac{M \hbar \omega}{2} \left\langle \eta^2 + \eta^{+2} - \eta \eta^{+} - \eta^{+} \eta \right\rangle$$

$$= \frac{M \hbar \omega}{2} \left\langle \eta \eta^{+} \right\rangle = \frac{M \hbar \omega}{2} \left\langle 1 + \eta^{+} \eta \right\rangle$$

$$= \frac{M \hbar \omega}{2}$$

Displacement operator:

$$D(v) \equiv e^{v a^{+} - v^{*} a}$$

$$e^{A+B} = e^{A} e^{B} e^{-[A,B]/2}$$

→ $$D(v) = e^{v a^{+}} e^{-v^{*} a} e^{-|v|^2/2} = e^{-v^{*} a} e^{v a^{+}} e^{|v|^2/2}$$

$$D^{+}(v) = e^{v^{*} a - v a^{+}} = D(-v) = D^{-1}(v)$$

$$= e^{-v a^{+}} e^{v^{*} a} e^{-|v|^2/2} = e^{v^{*} a} e^{-v a^{+}} e^{|v|^2/2}$$

$$\frac{\partial}{\partial v} D(v) = \left(a^{+} - v^{*}\right) D(v) = D(v) \left(a^{+} + v^{*}\right)$$

$$\frac{\partial}{\partial v} D^{+}(v) = -\left(a^{+} + v^{*}\right) D^{+}(v) = D^{+}(v) \left(-a^{+} + v^{*}\right)$$

→ $$\frac{\partial}{\partial v} [D^{+}(v)\, a\, D(v)] = D^{+}(v) \left[\left(-a^{+} + v^{*}\right) a + a\left(a^{+} - v^{*}\right) \right] D(v)$$

$$= D^{+}(v) [a, a^{+}] D(v)$$

$$= D^{+}(v) D(v) = 1$$

∴ $$\int_0^v d v \frac{\partial}{\partial v} [D^{+}(v)\, a\, D(v)] = D^{+}(v)\, a\, D(v) - D^{+}(0)\, a\, D(0)$$

$$= D^{+}(v)\, a\, D(v) - a$$

$$= \int_0^v d v = v$$

→ $$D^{+}(v)\, a\, D(v) = a + v$$

$$\frac{\partial}{\partial v^{*}} D(v) = -D(v) (a + v) = (-a + v) D(v)$$

$$\frac{\partial}{\partial v^{*}} D^{+}(v) = D^{+}(v) (a - v) = (a + v) D^{+}(v)$$

→ $$\frac{\partial}{\partial v^{*}} [D^{+}(v)\, a^{+} D(v)] = D^{+}(v) [(a - v) a^{+} + a^{+}(-a + v)] D(v)$$

$$= D^{+}(v) [a, a^{+}] D(v)$$

$$= D^{+}(v) D(v) = 1$$

∴ $$\int_0^{v^{*}} d v^{*} \frac{\partial}{\partial v^{*}} [D^{+}(v)\, a^{+} D(v)] = D^{+}(v)\, a^{+} D(v) - a^{+}$$

$$= \int_0^{v^{*}} d v^{*} = v^{*}$$

→ $$D^{+}(v)\, a^{+} D(v) = a^{+} + v^{*}$$

$$a D(v) \mid 0 \rangle = D(v) D^{+}(v)\, a\, D(v) \mid 0 \rangle$$

$$= D(v) (a + v) \mid 0 \rangle$$

$$= v\, D(v)\, |\, 0\rangle$$

Also $\langle 0\,|\, D^{+}(v)\, D(v)\,|\,0\rangle = \langle 0\,|\,0\rangle = 1$

∴ We can set $|\, v\rangle = D(v)\,|\,0\rangle$

$$a\,|\,0\rangle = 0 \qquad \langle 0\,|\, a^{+} = 0$$

→ $e^{\alpha a}\,|\,0\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} a^n \,|\,0\rangle = |\,0\rangle \qquad \forall\ \alpha = \text{const.}$

$$\langle 0\,|\, e^{\alpha a^{+}} = \langle 0\,|$$

$$|\, v\rangle = e^{v a^{+}}\, e^{-v^{*} a}\, e^{-|v|^2/2}\,|\,0\rangle$$
$$= e^{-|v|^2/2}\, e^{v a^{+}}\,|\,0\rangle$$
$$= e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{n!} a^{+n}\,|\,0\rangle$$
$$= e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{n!}}\,|\,n\rangle$$

where

$$a^{+} a = N \qquad \& \qquad N\,|\,n\rangle = n\,|\,n\rangle$$

$$\langle m\,|\,n\rangle = \delta_{m n}$$

→ $\langle m\,|\,v\rangle = e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{n!}} \langle m\,|\,n\rangle$

$$= e^{-|v|^2/2} \frac{v^m}{\sqrt{m!}}$$

$$|\,\langle n\,|\,v\rangle\,|^2 = e^{-|v|^2} \frac{|\,v\,|^{2n}}{n!}$$
$$= e^{-\bar{n}} \frac{\bar{n}^n}{n!} \qquad \bar{n} = \langle v\,|\,N\,|\,v\rangle = |\,v\,|^2$$

∴ $P_n = |\,\langle n\,|\,v\rangle\,|^2$ is a Poisson distribution.

$\{\,|\,v\rangle\}$ forms a non-orthogonal over-complete basis.

$$\langle u\,|\,v\rangle = e^{-|u|^2/2 - |v|^2/2} \langle 0\,|\, e^{-u a^{+}}\, e^{u^{*} a}\, e^{v a^{+}}\, e^{-v^{*} a}\,|\,0\rangle$$
$$= e^{-|u|^2/2 - |v|^2/2} \langle 0\,|\, e^{u^{*} a}\, e^{v a^{+}}\,|\,0\rangle$$

$$e^{A+B} = e^{A}\, e^{B}\, e^{-[A,B]/2}$$

→ $e^{u^{*} a}\, e^{v a^{+}} = e^{u^{*} a + v a^{+}}\, e^{u^{*} v/2} = e^{v a^{+}}\, e^{u^{*} a}\, e^{u^{*} v}$

∴ $\langle u\,|\,v\rangle = e^{-|u|^2/2 - |v|^2/2 + u^{*} v} \langle 0\,|\, e^{v a^{+}}\, e^{u^{*} a}\,|\,0\rangle$

$$= e^{-|u|^2/2 - |v|^2/2 + u^{*} v}$$

i.e., $\{\,|\,v\rangle\}$ is non-orthogonal.

$$\langle m\,|\,v\rangle = e^{-|v|^2/2} \frac{v^m}{\sqrt{m!}}$$

→ $\int d^2 v\, \langle n\,|\,v\rangle\langle v\,|\,m\rangle = \int d^2 v\, e^{-|v|^2} \frac{v^{*n}}{\sqrt{n!}} \frac{v^m}{\sqrt{m!}}$

$$= \frac{1}{\sqrt{n!\,m!}} \int_0^\infty dr\, e^{-r^2}\, r^{n+m+1} \int_0^{2\pi} d\theta\, e^{i(m-n)\theta} \qquad (v = r\, e^{i\theta})$$
$$= \frac{1}{\sqrt{n!\,m!}} \int_0^\infty dr\, e^{-r^2}\, r^{n+m+1}\, 2\pi\, \delta_{mn}$$
$$= \delta_{mn} \frac{2\pi}{n!} \int_0^\infty dr\, e^{-r^2}\, r^{2n+1}$$
$$= \pi\, \delta_{mn}$$

$$\therefore \qquad \frac{1}{\pi} \int d^2 v\; |\, v \rangle \langle v\, | = I$$

i.e., $\{ |\, v \rangle \}$ is complete.

Since the countable set $\{ |\, n \rangle \}$ is already complete, the set $\{ |\, v \rangle \}$, with v complex, must be over-complete. Thus, a countable subset of $\{ |\, v \rangle \}$ can span the eigen-space of H. Such a subset of v forms a lattice in the complex (or xy) plane.

Consider the square lattice points with lattice spacing $\sqrt{\pi}\; l$:

$$v_{mn} = \sqrt{\pi}\; l\, (m + i\, n)$$

$$\rightarrow \qquad \langle q \rangle = \langle v_{mn} \,|\, q \,|\, v_{mn} \rangle = \sqrt{\frac{\hbar}{2 M \omega}}\, (v_{mn} + v_{mn}^*) = \sqrt{\frac{2\pi\hbar}{M\omega}}\; l\, m$$

$$\langle p \rangle = \langle v_{mn} \,|\, p \,|\, v_{mn} \rangle = -i \sqrt{\frac{M \hbar \omega}{2}}\, (v_{mn} - v_{mn}^*) = \sqrt{2\pi M \hbar \omega}\; l\, n$$

$\therefore$ Phase space area of each state is

$$\left(\sqrt{\frac{2\pi\hbar}{M\omega}}\; l \right) \left(\sqrt{2\pi M \hbar \omega}\; l \right) = 2\pi\hbar\, l^2$$

It can be proved that $\{ |\, v_{mn} \rangle \}$ is complete if $l \le 1$.
In particular, only one dependent state, satisfying $\sum_{m,n} |\, v_{mn} \rangle = 0$, exists for $l = 1$.

Another minimal complete set is given by the von Neumann lattice:

$$v_{mn} = \sqrt{\pi}\, \left(l_q\, m + l_p\, n \right)$$

with $\quad l_j = l_j^x + i\, l_j^y \qquad$ for $j = q, p$