1.6. Squeezed Coherent State

Bogoliubov transformation:

$$\zeta = \eta \cosh\tau + \eta^+ e^{i\beta} \sinh\tau$$

$$\zeta^+ = \eta^+ \cosh\tau + \eta e^{-i\beta} \sinh\tau \qquad (\beta,\ \tau \text{ real})$$

$$\zeta \cosh\tau = \eta \cosh^2\tau + \eta^+ e^{i\beta} \cosh\tau \sinh\tau$$

$$\zeta^+ e^{i\beta} \sinh\tau = \eta^+ e^{i\beta} \sinh\tau \cosh\tau + \eta \sinh^2\tau$$

→ Inverse transform:

$$\eta = \zeta \cosh\tau - \zeta^+ e^{i\beta} \sinh\tau$$

$$\eta^+ = \zeta^+ \cosh\tau - \zeta e^{-i\beta} \sinh\tau$$

In terms of generator G, we have

$$\zeta = e^{-iG} \eta e^{iG}$$

where

$$G = \frac{i}{2} \tau \left(e^{-i\beta} \eta^2 - e^{i\beta} \eta^{+2} \right)$$

$$G^+ = -\frac{i}{2} \tau \left(e^{i\beta} \eta^{+2} - e^{-i\beta} \eta^2 \right) = G$$

so that the transformation is unitary &

$$\zeta^+ = e^{-iG} \eta^+ e^{iG}$$

Proof:

$$[-iG, \eta] = \left[-\frac{1}{2} \tau e^{i\beta} \eta^{+2}, \eta \right] = \tau e^{i\beta} \eta^+$$

$$[-iG, [-iG, \eta]] = \left[\frac{1}{2} \tau e^{-i\beta} \eta^2, \tau e^{i\beta} \eta^+ \right] = \tau^2 \eta$$

$$[-iG, [-iG, [-iG, \eta]]] = \left[-\frac{1}{2} \tau e^{i\beta} \eta^{+2}, \tau^2 \eta \right] = \tau^3 e^{i\beta} \eta^+$$

$$[-iG, [-iG, [-iG, \eta]]] = \left[\frac{1}{2} \tau e^{-i\beta} \eta^2, \tau^3 e^{i\beta} \eta^+ \right] = \tau^4 \eta$$

→ $$\zeta = \eta + [-iG, \eta] + \frac{1}{2!}[-iG, [-iG, \eta]] + \dots$$

$$= \eta + \frac{1}{2!} \tau^2 \eta + \frac{1}{4!} \tau^4 \eta + \dots + \tau e^{i\beta} \eta^+ + \frac{1}{3!} \tau^3 e^{i\beta} \eta^+ + \dots$$

$$= \eta \cosh\tau + \eta^+ e^{i\beta} \sinh\tau$$

Let $| v \rangle\rangle = e^{-iG} \left| v \right\rangle$

$$a = v + \eta$$

→ $$b = e^{-iG} a e^{iG} = v + e^{-iG} \eta e^{iG} = v + \zeta$$

∴ $$b \,| v \rangle\rangle = b e^{-iG} \left| v \right\rangle = e^{-iG} a \left| v \right\rangle = e^{-iG} v \left| v \right\rangle = v \,| v \rangle\rangle$$

& $$\zeta \,| v \rangle\rangle = 0$$

Thus, $| v \rangle\rangle$ is the normalized coherent (eigen) state of b and the Fock vacuum of ζ.

$$\eta = \zeta \cosh\tau - \zeta^+ e^{i\beta} \sinh\tau$$

$$\eta^+ = \zeta^+ \cosh\tau - \zeta e^{-i\beta} \sinh\tau$$

$$\Delta \mathrm{q} = \sqrt{\frac{\hbar}{2 M \omega}} \left(\eta^{+} + \eta\right)$$

$$= \sqrt{\frac{\hbar}{2 M \omega}} \left[\left(\cosh \tau - e^{-i \beta} \sinh \tau\right) \zeta + \left(\cosh \tau - e^{i \beta} \sinh \tau\right) \zeta^{+}\right]$$

$$\Delta \mathrm{p} = i \sqrt{\frac{M \hbar \omega}{2}} \left(\eta^{+} - \eta\right)$$

$$= i \sqrt{\frac{M \hbar \omega}{2}} \left[-\left(\cosh \tau + e^{-i \beta} \sinh \tau\right) \zeta + \left(\cosh \tau + e^{i \beta} \sinh \tau\right) \zeta^{+}\right]$$

$$\left\langle\left\langle (\Delta q)^{2} \right\rangle\right\rangle = \frac{\hbar}{2 M \omega} \left\langle\left\langle v \mid \left(\eta^{+} + \eta\right)^{2} \mid v \right\rangle\right\rangle$$
$$= \frac{\hbar}{2 M \omega} \left(\cosh \tau - e^{-i \beta} \sinh \tau\right)\left(\cosh \tau - e^{i \beta} \sinh \tau\right) \left\langle\left\langle v \mid \zeta \zeta^{+} \mid v \right\rangle\right\rangle$$
$$= \frac{\hbar}{2 M \omega} \left(\cosh \tau - e^{-i \beta} \sinh \tau\right)\left(\cosh \tau - e^{i \beta} \sinh \tau\right)$$

$$\left\langle\left\langle (\Delta p)^{2} \right\rangle\right\rangle = -\frac{M \hbar \omega}{2} \left\langle\left\langle v \mid \left(\eta^{+} - \eta\right)^{2} \mid v \right\rangle\right\rangle$$
$$= \frac{M \hbar \omega}{2} \left(\cosh \tau + e^{-i \beta} \sinh \tau\right)\left(\cosh \tau + e^{i \beta} \sinh \tau\right) \left\langle\left\langle v \mid \zeta \zeta^{+} \mid v \right\rangle\right\rangle$$
$$= \frac{M \hbar \omega}{2} \left(\cosh \tau + e^{-i \beta} \sinh \tau\right)\left(\cosh \tau + e^{i \beta} \sinh \tau\right)$$

$$\left\langle\left\langle (\Delta q)^{2} \right\rangle\right\rangle \left\langle\left\langle (\Delta p)^{2} \right\rangle\right\rangle = \frac{\hbar^{2}}{4} \left(\cosh^{2} \tau - e^{-2 i \beta} \sinh^{2} \tau\right)\left(\cosh^{2} \tau - e^{2 i \beta} \sinh^{2} \tau\right)$$

Setting $\beta = 0$, or $\pm \pi$ gives

$$\left\langle\left\langle (\Delta q)^{2} \right\rangle\right\rangle \left\langle\left\langle (\Delta p)^{2} \right\rangle\right\rangle = \frac{\hbar^{2}}{4} \qquad \text{(minimal uncertainty; squeezed state)}$$

For $\beta = 0$,

$$\zeta = \eta \cosh \tau + \eta^{+} \sinh \tau$$
$$\zeta^{+} = \eta^{+} \cosh \tau + \eta \sinh \tau$$

$$\eta = \zeta \cosh \tau - \zeta^{+} \sinh \tau$$
$$\eta^{+} = \zeta^{+} \cosh \tau - \zeta \sinh \tau$$

$$\Delta \mathrm{q} = \sqrt{\frac{\hbar}{2 M \omega}} (\cosh \tau - \sinh \tau) \left(\zeta + \zeta^{+}\right) = \sqrt{\frac{\hbar}{2 M \omega}} e^{-\tau} \left(\zeta^{+} + \zeta\right)$$

$$\Delta \mathrm{p} = i \sqrt{\frac{M \hbar \omega}{2}} (\cosh \tau + \sinh \tau) \left(\zeta^{+} - \zeta\right) = i \sqrt{\frac{M \hbar \omega}{2}} e^{\tau} \left(\zeta^{+} - \zeta\right)$$

$$\zeta = \sqrt{\frac{M \omega}{2 \hbar}} e^{\tau} \Delta \mathrm{q} + i \sqrt{\frac{1}{2 M \hbar \omega}} e^{-\tau} \Delta \mathrm{p}$$

$$\zeta^+ = \sqrt{\frac{M\omega}{2\hbar}}\, e^{\tau}\,\Delta q - i\sqrt{\frac{1}{2M\hbar\omega}}\, e^{-\tau}\,\Delta p$$

$$\zeta^+ + \zeta = (\eta + \eta^+)(\cosh\tau + \sinh\tau) = (\eta + \eta^+)\, e^{\tau}$$

$$\zeta^+ - \zeta = (-\eta + \eta^+)(\cosh\tau - \sinh\tau) = (-\eta + \eta^+)\, e^{-\tau}$$

→ $$\Delta q = \sqrt{\frac{\hbar}{2M\omega}}\,(\eta + \eta^+)$$

$$\Delta p = i\sqrt{\frac{M\hbar\omega}{2}}\,(-\eta + \eta^+)$$

∴ $$\eta = \sqrt{\frac{M\omega}{2\hbar}}\,\Delta q + i\sqrt{\frac{1}{2M\hbar\omega}}\,\Delta p$$

$$\eta^+ = \sqrt{\frac{M\omega}{2\hbar}}\,\Delta q - i\sqrt{\frac{1}{2M\hbar\omega}}\,\Delta p$$

$$\langle\!\langle (\Delta q)^2 \rangle\!\rangle = \frac{\hbar}{2M\omega}(\cosh\tau - \sinh\tau)(\cosh\tau - \sinh\tau) = \frac{\hbar}{2M\omega}\, e^{-2\tau}$$

$$\langle\!\langle (\Delta p)^2 \rangle\!\rangle = \frac{M\hbar\omega}{2}(\cosh\tau + \sinh\tau)(\cosh\tau + \sinh\tau) = \frac{M\hbar\omega}{2}\, e^{2\tau}$$

$$\langle\!\langle (\Delta q)^2 \rangle\!\rangle \langle\!\langle (\Delta p)^2 \rangle\!\rangle = \frac{\hbar^2}{4}$$

$$\eta = \zeta\cosh\tau - \zeta^+\sinh\tau \qquad \zeta \mid v\rangle\!\rangle = 0$$

→ $$\langle\!\langle v \mid \eta \mid v\rangle\!\rangle = 0$$

∴ $$\langle\!\langle v \mid a \mid v\rangle\!\rangle = v + \langle\!\langle v \mid \eta \mid v\rangle\!\rangle = v$$

Hence, $\mid v\rangle\!\rangle$ is also called the squeezed coherent state even though it’s not the coherent state of a.

Summary

For a quadratic Hamiltonian,

$$H = \frac{1}{2M}p^2 + \frac{1}{2}M\omega^2 q^2 = \hbar\omega\left(a^+ a + \frac{1}{2}\right)$$

where

$$a = \frac{1}{\sqrt{2M\hbar\omega}}(M\omega q + i p) \qquad a^+ = \frac{1}{\sqrt{2M\hbar\omega}}(M\omega q - i p)$$

$$q = \sqrt{\frac{\hbar}{2M\omega}}\,(a + a^+) \qquad p = -i\sqrt{\frac{M\hbar\omega}{2}}\,(a - a^+)$$

η takes the coherent state of a as vacuum :

$$a \mid v\rangle = v \mid v\rangle$$

$$a = v + \eta$$

$$\eta \mid v \rangle = 0$$

ζ is the Bogoliubov transform of η :

$$\zeta = \eta \cosh \tau + \eta^{+} \sinh \tau$$

$$\zeta^{+} = \eta^{+} \cosh \tau + \eta \sinh \tau \qquad (\tau \text{ real})$$

In terms of a unitary transformation:

$$\zeta = e^{-i G} \eta e^{i G}$$

with

$$G = \frac{i}{2} \tau \left(\eta^2 - \eta^{+2} \right) = G^{+}$$

Vacuum of ζ is the coherent state of $b = v + \zeta$:

$$\mid v \rangle\rangle = e^{-i G} \left| v \right\rangle$$

$$\zeta \mid v \rangle\rangle = 0$$

$$b \mid v \rangle\rangle = v \mid v \rangle\rangle$$

$\mid v \rangle\rangle$ is a normalized squeezed state with minimal uncertainty:

$$\langle\langle (\Delta q)^2 \rangle\rangle = \frac{\hbar}{2 M \omega} e^{-2 \tau}$$

$$\langle\langle (\Delta p)^2 \rangle\rangle = \frac{M \hbar \omega}{2} e^{2 \tau}$$

$$\langle\langle (\Delta q)^2 \rangle\rangle \langle\langle (\Delta p)^2 \rangle\rangle = \frac{\hbar^2}{4}$$

In case $\mid v \rangle\rangle$ is not normalized so that $\langle\langle v \mid v \rangle\rangle = f$, we have

$$\langle\langle (\Delta q)^2 \rangle\rangle = \frac{\hbar}{2 M \omega f} e^{-2 \tau}$$

$$\langle\langle (\Delta p)^2 \rangle\rangle = \frac{M \hbar \omega}{2 f} e^{2 \tau}$$

$$\langle\langle (\Delta q)^2 \rangle\rangle \langle\langle (\Delta p)^2 \rangle\rangle = \frac{\hbar^2}{4 f^2}$$