1.7. Particle Number And Phase

Let

$$a = e^{i\Theta}\sqrt{N} \qquad\qquad a^+ = \sqrt{N}\, e^{-i\Theta}$$

where

$$\rightarrow \qquad a^+ a = N \qquad\qquad a\,a^+ = e^{i\Theta} N e^{-i\Theta}$$

$$\begin{aligned}[a, a^+] &= e^{i\Theta} N e^{-i\Theta} - N \\ &= \left(e^{i\Theta} N - N e^{i\Theta}\right) e^{-i\Theta} \\ &= \left[e^{i\Theta}, N\right] e^{-i\Theta} \\ &\equiv 1\end{aligned}$$

$$\rightarrow \qquad \left[e^{i\Theta}, N\right] = e^{i\Theta}$$

To lowest order in Θ on both sides,

$$[\,i\,\Theta, N] = 1$$

$$\rightarrow \qquad [\,N, \Theta] = i$$

Comparing with

$$[x, p] = i\,\hbar \qquad \rightarrow \qquad \Delta x\,\Delta p \geq \frac{\hbar}{2}$$

we have,

$$\Delta n\,\Delta\theta \geq \frac{1}{2}$$

where

$$\Delta n = \sqrt{\left\langle(\Delta N)^2\right\rangle} \qquad\qquad \Delta\theta = \sqrt{\left\langle(\Delta\Theta)^2\right\rangle}$$

$$a \mid n\rangle = \sqrt{n}\,\left|\, n-1\right\rangle \quad a^+ \mid n\rangle = \sqrt{n+1}\,\left|\, n+1\right\rangle$$

$$\rightarrow \qquad a^+ a \mid n\rangle = \sqrt{n}\, a^+ \left|\, n-1\right\rangle = n \mid n\rangle = N \mid n\rangle$$

$$\begin{aligned}\left\langle m \,\middle|\, \left[e^{i\Theta}, N\right] \,\middle|\, n\right\rangle &= \left\langle m \mid \left(e^{i\Theta} n - m\, e^{i\Theta}\right) \middle|\, n\right\rangle \\ &= (n-m)\left\langle m \,\middle|\, e^{i\Theta} \,\middle|\, n\right\rangle \\ &= \left\langle m \mid e^{i\Theta} \,\middle|\, n\right\rangle\end{aligned}$$

$$\rightarrow \qquad \left\langle m \mid e^{i\Theta} \,\middle|\, n\right\rangle = 0 \qquad \forall\;\; n \neq m+1$$

and $\quad e^{i\Theta}\left|\, n\right\rangle = c_n \mid n-1\rangle \quad$ where c_n is a constant.

Thus, $e^{i\Theta}$ acts like a.

Caution:

It's natural to assume

$$e^{i\Theta} e^{-i\Theta} = e^{-i\Theta} e^{i\Theta} = I$$

However, since

$$e^{i\Theta}\left|\, 0\right\rangle = 0$$

we have

$$e^{-i\Theta}\left(e^{i\Theta}\left|\, 0\right\rangle\right) = 0$$

Thus, we have

$$e^{i\Theta} e^{-i\Theta} = e^{-i\Theta} e^{i\Theta} + \left|\, 0\right\rangle\langle 0 \mid = I$$

$$\left\langle n \,\right| e^{-i\Theta} = \langle n-1 \,|\; c_n^*$$

→ $$\left\langle n \,\right| e^{-i\Theta} e^{i\Theta} \left| \, n \right\rangle = c_n^* c_n \langle n-1 \mid n-1 \rangle$$

∴ $$c_n^* c_n = 1 \qquad \forall\, n > 0$$

Setting c_n real, we have

$$e^{i\Theta} \left| \, n \right\rangle = \left| \, n-1 \right\rangle$$

$$\left\langle n \right| e^{i\Theta} \left| \, n \right\rangle = \langle n \mid n-1 \rangle = 0$$

From

$$[x, p] = i\,\hbar \qquad \to \qquad p = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

we have

$$[N, \Theta] = i \qquad \to \qquad \Theta = -i \frac{\partial}{\partial n}$$

Let

$$\Theta \mid \theta \rangle = \theta \mid \theta \rangle$$

$$\mid \theta \rangle = \sum_n c_n(\theta) \left| \, n \right\rangle$$

→ $$\Theta \mid \theta \rangle = -i \sum_n \frac{\partial\, c_n(\theta)}{\partial\, n} \left| \, n \right\rangle$$

$$= \theta \,|\theta \;\rangle \; = \theta \sum_n c_n(\theta) \left| \, n \right\rangle$$

→ $$-i \frac{\partial\, c_n(\theta)}{\partial\, n} = \theta\, c_n(\theta)$$

$$\frac{\partial \ln c_n(\theta)}{\partial\, n} = i\,\theta$$

$$c_n(\theta) = C\, e^{i\,\theta\, n}$$

∴ $$\mid \theta \rangle = C \sum_n e^{i\,\theta\, n} \left| \, n \right\rangle$$

$$\langle \theta \mid = C^* \sum_n \left\langle n \,\right| e^{-i\,\theta\, n}$$

$$\langle \theta' \mid \theta \rangle = |\, C\, |^2 \sum_{m,n} e^{-i\,\theta'\, m + i\,\theta\, n} \langle\, m \mid n \,\rangle$$

$$= |\, C\, |^2 \sum_n e^{i\,(\theta - \theta')\, n}$$

$$= 2\,\pi\; |\, C\, |^2\, \delta(\theta - \theta')$$

$$\equiv 2\,\pi\, \delta(\theta - \theta')$$

→ $$|\, C\, |^2 = 1$$

Setting C real gives

$$\mid \theta \rangle = \sum_n e^{i\,\theta\, n} \left| \, n \right\rangle$$

$$\mid n \rangle = \int_0^{2\pi} \frac{d\,\theta}{2\,\pi}\, e^{-i\,\theta\, n} \left| \; \theta \right\rangle$$