1.8. Macroscopic Coherence

Characteristics of a coherent state

$$\langle v \mid a \mid v \rangle = v$$

$$\langle v \mid N \mid v \rangle = \langle v \mid a^+ a \mid v \rangle = \mid v \mid^2 = n$$

$$a = v + \eta \qquad\qquad a^+ = v^* + \eta^+$$

where

$$a \mid v \rangle = v \mid v \rangle = \sqrt{n}\, e^{i\theta} \left| v \right\rangle \qquad\qquad v = \sqrt{n}\, e^{i\theta}$$

$$\rightarrow \qquad \eta \mid v \rangle = 0 \qquad\qquad \langle v \mid \eta^+ = 0$$

$$\rightarrow \qquad N = a^+ a = \left(v^* + \eta^+\right)(v + \eta)$$

$$= n + \eta^+ \eta + v^* \eta + v\, \eta^+$$

$$N \mid v \rangle = \left(n + v\, \eta^+\right) \mid v \rangle \qquad \rightarrow \qquad (N - n) \mid v \rangle = v\, \eta^+ \mid v \rangle$$

$$\langle v \mid N = \langle v \mid (n + v^* \eta) \quad \rightarrow \quad \langle v \mid (N - n) = \langle v \mid v^* \eta$$

$$(\Delta n)^2 = \left\langle (\Delta N)^2 \right\rangle = \left\langle v \mid (N - n)^2 \mid v \right\rangle$$

$$= \mid v \mid^2 \left\langle v \mid \eta\, \eta^+ \mid v \right\rangle$$

$$= n \left\langle v \mid 1 + \eta^+ \eta \mid v \right\rangle$$

$$= n$$

From

$$N \mid v \rangle = \left(n + v\, \eta^+\right) \mid v \rangle$$

we get

$$N^{-1/2} \left| v \right\rangle = \left(n + v\, \eta^+\right)^{-1/2} \left| v \right\rangle$$

$$= \frac{1}{\sqrt{n}} \left(1 + \frac{1}{v^*} \eta^+\right)^{-1/2} \left| v \right\rangle$$

$$a = e^{i\Theta} \sqrt{N}$$

$$\rightarrow \qquad e^{i\Theta} \left| v \right\rangle = a\, N^{-1/2} \left| v \right\rangle$$

$$= \frac{v}{\sqrt{n}} \left(1 + \frac{\eta}{v}\right) \left(1 + \frac{1}{v^*} \eta^+\right)^{-1/2} \left| v \right\rangle$$

$$= \frac{v}{\sqrt{n}} \left(1 + \frac{\eta}{v}\right) [1 + f(\eta^+)] \left| v \right\rangle$$

where

$$f(\eta^+) = -\frac{1}{2 v^*} \eta^+ + \frac{3}{8} \left(\frac{1}{v^*} \eta^+\right)^2 - \dots$$

$$\therefore \qquad e^{i\Theta} \left| v \right\rangle = \frac{v}{\sqrt{n}} \left(1 + \frac{\eta}{v} + f(\eta^+) + \frac{\eta}{v} f(\eta^+)\right] \left| v \right\rangle$$

$$= \frac{v}{\sqrt{n}} \left(1 + f(\eta^+) + \frac{\eta}{v} f(\eta^+)\right] \left| v \right\rangle$$

$$\left\langle v \right| (\eta^+)^k = 0$$

$$\rightarrow \qquad \langle v \mid e^{i\Theta} \mid v \rangle = \frac{v}{\sqrt{n}} \left\langle v \,\middle|\, \left[1 + \frac{\eta}{v} f(\eta^{+}) \right] \,\middle|\, v \right\rangle$$

$$\eta\, \eta^{+k} = (\eta^{+}\, \eta + 1)\, (\eta^{+})^{k-1}$$

$$\rightarrow \qquad \langle v \mid \eta\, \eta^{+k} \mid v \rangle = \begin{cases} 0 & k \geq 2 \\ 1 & k = 1 \end{cases}$$

$$\therefore \qquad \langle v \mid e^{i\Theta} \mid v \rangle = \frac{v}{\sqrt{n}} \left\langle v \,\middle|\, \left(1 - \frac{1}{2\, n} \eta \eta^{+} \right) \,\middle|\, v \right\rangle$$

$$= e^{i\,\theta} \left(1 - \frac{1}{2\, n} \right)$$

$$\rightarrow e^{i\,\theta} \;\; \text{for } n \rightarrow \infty$$