3.2. Schrodinger Field

Schrodinger eq $$i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \varphi + V\varphi$$

Lagrangian

$$\mathcal{L} = i\hbar\phi^* \partial_t \phi - \frac{\hbar^2}{2m}\nabla\phi^* \cdot \nabla\phi - V\phi^*\phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = 0 \qquad \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\nabla \phi^*)} = -\frac{\hbar^2}{2m}\nabla\phi \qquad \nabla\cdot\frac{\partial \mathcal{L}}{\partial(\nabla \phi^*)} = -\frac{\hbar^2}{2m}\nabla^2\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi^*} = i\hbar\partial_t\phi - V\phi$$

→ $$-\frac{\hbar^2}{2m}\nabla^2\phi - i\hbar\partial_t\phi + V\phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = i\hbar\phi^* \qquad \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = i\hbar\partial_t\phi^*$$

$$\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} = -\frac{\hbar^2}{2m}\nabla\phi^* \qquad \nabla\cdot\frac{\partial \mathcal{L}}{\partial(\nabla \phi)} = -\frac{\hbar^2}{2m}\nabla^2\phi^*$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -V\phi^*$$

→ $$i\hbar\partial_t\phi^* - \frac{\hbar^2}{2m}\nabla^2\phi^* + V\phi^* = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = i\hbar\phi^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi^*)} = 0 \neq (\pi)^*$$

$$\mathcal{H} = i\hbar\phi^*\partial_t\phi - \mathcal{L} = \frac{\hbar^2}{2m}\nabla\phi^*\cdot\nabla\phi + V\phi^*\phi$$

2nd Quantization via equal-time commutation relations :

$$[\phi(t,\mathbf{r}), \pi(t,\mathbf{r}')]_{\mp} = i\hbar\,\delta(\mathbf{r}-\mathbf{r}') \quad \text{for} \quad \begin{matrix}\text{bosons}\\ \text{fermions}\end{matrix}$$

→ $$[\phi(t,\mathbf{r}), \phi^+(t,\mathbf{r}')]_{\mp} = \delta(\mathbf{r}-\mathbf{r}')$$

Hamiltonian

Let

$$H\varphi_j(\mathbf{r}) = \epsilon_j\varphi_j(\mathbf{r}) \quad \text{with} \quad \langle \varphi_j \mid \varphi_k \rangle = \delta_{jk}$$

→ $$\phi(t,\mathbf{r}) = \sum_j a_j\varphi_j(\mathbf{r})\,e^{-i\epsilon_j t/\hbar} \qquad \phi^+(t,\mathbf{r}) = \sum_j a_j^+\varphi_j^*(\mathbf{r})\,e^{i\epsilon_j t/\hbar}$$

$$\nabla\phi = \sum_j a_j\,(\nabla\varphi_j)\,e^{-i\,\epsilon_j\,t/\hbar} \qquad\qquad \nabla\phi^+ = \sum_j a_j^+\,(\nabla\varphi_j^*)\,e^{i\,\epsilon_j\,t/\hbar}$$

$$\mathcal{H} = \sum_{j,k}\left[\frac{\hbar^2}{2\,m}\,(\nabla\varphi_j^*\cdot\nabla\varphi_k) + V\,\varphi_j^*\,\varphi_k\right] a_j^+\,a_k\,e^{-i\,(\epsilon_k-\epsilon_j)\,t/\hbar}$$

$$\begin{aligned}
H &= \int d^3r\,\mathcal{H} \\
&= \sum_{j,k} a_j^+\,a_k\,e^{-i\,(\epsilon_k-\epsilon_j)\,t/\hbar}\int d^3r\left[\frac{\hbar^2}{2\,m}\,(\nabla\varphi_j^*\cdot\nabla\varphi_k) + V\,\varphi_j^*\,\varphi_k\right] \\
&= \sum_{j,k} a_j^+\,a_k\,e^{-i\,(\epsilon_k-\epsilon_j)\,t/\hbar}\int d^3r\,\varphi_j^*\left(-\frac{\hbar^2}{2\,m}\,\nabla^2\varphi_k + V\,\varphi_k\right) \\
&= \sum_{j,k} a_j^+\,a_k\,e^{-i\,(\epsilon_k-\epsilon_j)\,t/\hbar}\int d^3r\,\varphi_j^*\,H\,\varphi_k \\
&= \sum_{j,k} a_j^+\,a_k\,e^{-i\,(\epsilon_k-\epsilon_j)\,t/\hbar}\,\epsilon_j\,\delta_{j\,k} \\
&= \sum_j \epsilon_j\,a_j^+\,a_j
\end{aligned}$$

Momentum Expansion

See FourierTransforms.pdf .

For interacting particles,

$$\mathcal{L} = i\,\hbar\,\phi^+(x)\,\partial_t\,\phi(x) - \frac{\hbar^2}{2\,m}\,\nabla\phi^+(x)\cdot\nabla\phi(x) - V(\boldsymbol{x})\;\;\phi^+(x)\,\phi(x) - \frac{1}{2}\int d^3r'\;\phi^+(x)\,\phi^+(x')\,U(\boldsymbol{r}-\boldsymbol{r}')\,\phi(x')\,\phi(x)\Big|_{t=t'}$$

where $\quad x = (t,\,\boldsymbol{r}) \qquad x' = (t',\,\boldsymbol{r}')$

$$\rightarrow \qquad \mathcal{H} = \frac{\hbar^2}{2\,m}\,\nabla\phi^+(x)\cdot\nabla\phi(x) + V(\boldsymbol{r})\;\;\phi^+(x)\,\phi(x) + \frac{1}{2}\int d^3r'\,\phi^+(x)\,\phi^+(x')\,U(\boldsymbol{r}-\boldsymbol{r}')\,\phi(x')\,\phi(x)\Big|_{t=t'}$$

Momentum (plane wave) expansion:

$$\phi(x) = \int\frac{d^3k}{(2\,\pi)^{3/2}}\,a_{\boldsymbol{k}}(t)\;e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\int d^3r\,e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}} = (2\,\pi)^3\,\delta(\boldsymbol{k})$$

$$\rightarrow \qquad a_{\boldsymbol{k}}(t) = \int\frac{d^3r}{(2\,\pi)^{3/2}}\,\phi(x)\;e^{-i\,\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$[\,\phi(t,\,\boldsymbol{r}),\,\phi^+(t,\,\boldsymbol{r}')]_\mp = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

$$\rightarrow \qquad \int\frac{d^3k}{(2\,\pi)^{3/2}}\int\frac{d^3k'}{(2\,\pi)^{3/2}}\,[a_{\boldsymbol{k}}(t),\,a_{\boldsymbol{k}'}^+(t)]_\mp\;e^{i\,\boldsymbol{k}\cdot\boldsymbol{r}-i\,\boldsymbol{k}'\cdot\boldsymbol{r}'} = \int\frac{d^3k}{(2\,\pi)^3}\,e^{i\,\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

$$\rightarrow \qquad [a_{\boldsymbol{k}}(t),\,a_{\boldsymbol{k}'}^+(t)]_\mp = \delta_{\boldsymbol{k},\boldsymbol{k}'}$$

Similarly,

$$[\phi(t,\boldsymbol{r}),\ \phi(t,\boldsymbol{r}')]_\mp = [\phi^+(t,\boldsymbol{r}),\ \phi^+(t,\boldsymbol{r}')]_\mp = 0$$

$$\to \quad [a_{\boldsymbol{k}}(t),\ a_{\boldsymbol{k}'}(t)]_\mp = [a^+_{\boldsymbol{k}}(t),\ a^+_{\boldsymbol{k}'}(t)]_\mp = 0$$

Total Momentum & Hamiltonian

$$\boldsymbol{p} = \int d^3 r\, \phi^+(x) \frac{\hbar}{i} \nabla \phi(x)$$

$$= \int d^3 r \int \frac{d^3 k}{(2\pi)^{3/2}} \int \frac{d^3 k'}{(2\pi)^{3/2}}\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}'}(t)\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{\hbar}{i} \nabla\ e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}$$

$$= \int d^3 r \int \frac{d^3 k}{(2\pi)^{3/2}} \int \frac{d^3 k'}{(2\pi)^{3/2}}\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}'}(t)\, \hbar \boldsymbol{k}'\ e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}}$$

$$= \int d^3 k \int d^3 k'\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}'}(t)\, \hbar \boldsymbol{k}'\ \delta(\boldsymbol{k}'-\boldsymbol{k})$$

$$= \int d^3 k\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}}(t)\, \hbar \boldsymbol{k}$$

Similarly

$$\int d^3 r \frac{\hbar^2}{2m} \nabla\phi^+(x)\cdot\nabla\phi(x) = -\frac{\hbar^2}{2m}\int d^3 x\, \phi^+(x)\nabla^2\phi(x)$$

$$= \int d^3 k\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}}(t) \frac{\hbar^2 \boldsymbol{k}^2}{2m}$$

$$\int d^3 r\, V(\boldsymbol{x})\, \phi^+(x)\, \phi(x) = \int d^3 r \int \frac{d^3 k}{(2\pi)^{3/2}} \int \frac{d^3 k'}{(2\pi)^{3/2}}\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}'}(t)\, V(\boldsymbol{r})\ e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}$$

$$= \int d^3 k \int \frac{d^3 k'}{(2\pi)^3}\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}'}(t)\, V(\boldsymbol{k}-\boldsymbol{k}') \qquad f(\boldsymbol{k}) = \int d^3 r\, f(\boldsymbol{r})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$= \int d^3 k \int \frac{d^3 q}{(2\pi)^3}\ a^+_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}-\boldsymbol{q}}(t)\, V(\boldsymbol{q}) \qquad \boldsymbol{q} = \boldsymbol{k}-\boldsymbol{k}'$$

$$\frac{1}{2}\int d^3 r \int d^3 r'\, \phi^+(x)\, \phi^+(x')\, U(\boldsymbol{r}-\boldsymbol{r}')\, \phi(x')\, \phi(x)\Big|_{t=t'}$$

$$= \frac{1}{2}\int d^3 r \int d^3 r' \int \frac{d^3 k_1}{(2\pi)^{3/2}} \int \frac{d^3 k_2}{(2\pi)^{3/2}} \int \frac{d^3 k_3}{(2\pi)^{3/2}} \int \frac{d^3 k_4}{(2\pi)^{3/2}}$$

$$a^+_{\boldsymbol{k}_1}(t)\, a^+_{\boldsymbol{k}_2}(t)\, a_{\boldsymbol{k}_3}(t)\, a_{\boldsymbol{k}_4}(t)\, U(\boldsymbol{r}-\boldsymbol{r}')\, e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_4)\cdot\boldsymbol{r}-i(\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{r}'}$$

$$= \frac{1}{2}\int \frac{d^3 k_1}{(2\pi)^{3/2}} \int \frac{d^3 k_2}{(2\pi)^{3/2}} \int \frac{d^3 k_3}{(2\pi)^{3/2}} \int \frac{d^3 k_4}{(2\pi)^{3/2}}$$

$$a^+_{\boldsymbol{k}_1}(t)\, a^+_{\boldsymbol{k}_2}(t)\, a_{\boldsymbol{k}_3}(t)\, a_{\boldsymbol{k}_4}(t)\, U(\boldsymbol{k}_1-\boldsymbol{k}_4-\boldsymbol{k}_2+\boldsymbol{k}_3)$$

$$= \frac{1}{2}\int d^3 k \int d^3 k' \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3}\, a^+_{\boldsymbol{k}}(t)\, a^+_{\boldsymbol{k}'}(t)\, a_{\boldsymbol{k}'-\boldsymbol{q}'}(t)\, a_{\boldsymbol{k}-\boldsymbol{q}}(t)\, U(\boldsymbol{q}-\boldsymbol{q}')$$

where $\quad \boldsymbol{k} = \boldsymbol{k}_1 \qquad \boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_4$

$\qquad\quad \boldsymbol{k}' = \boldsymbol{k}_2 \qquad \boldsymbol{q}' = \boldsymbol{k}_2 - \boldsymbol{k}_3$

$$\to \quad H = \int d^3 r\, \mathcal{H}$$

$$= \int d^3 k \, \frac{\hbar^2 \mathbf{k}^2}{2m} a^+_{\mathbf{k}}(t)\, a_{\mathbf{k}}(t)$$

$$+ \int d^3 k \int \frac{d^3 q}{(2\pi)^3} V(\mathbf{q})\, a^+_{\mathbf{k}}(t)\, a_{\mathbf{k}-\mathbf{q}}(t)$$

$$+ \frac{1}{2} \int d^3 k \int d^3 k' \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 q'}{(2\pi)^3} U(\mathbf{q}-\mathbf{q}')\, a^+_{\mathbf{k}}(t)\, a^+_{\mathbf{k}'}(t)\, a_{\mathbf{k}'-\mathbf{q}'}(t)\, a_{\mathbf{k}-\mathbf{q}}(t)$$

Systems with Finite Volume

Momentum (plane wave) expansion:

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \tilde{a}_{\mathbf{k}}(t)\, e^{i\,\mathbf{k}\cdot\mathbf{r}} \qquad \mathbf{k} = \frac{2\pi}{V^{1/3}}(n_1, n_2, n_3)$$

$$\int d^3 r\, e^{i\,\mathbf{k}\cdot\mathbf{r}} = V\,\delta_{\mathbf{k},\mathbf{0}}$$

$$\tilde{a}_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}} \int d^3 r\, \phi(x)\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} \quad \rightarrow \quad \sum_{\mathbf{k}} e^{-i\,\mathbf{k}\cdot\mathbf{r}} = V\,\delta(\mathbf{r})$$

$$\phi(x) \approx \int \frac{d^3 k}{(2\pi)^{3/2}} a_{\mathbf{k}}(t)\, e^{i\,\mathbf{k}\cdot\mathbf{r}} \quad \rightarrow \quad \tilde{a}_{\mathbf{k}}(t) \approx \sqrt{\frac{(2\pi)^3}{V}}\, a_{\mathbf{k}}(t)$$

$$\left[\tilde{a}_{\mathbf{k}}(t),\ \tilde{a}^+_{\mathbf{k}'}(t)\right]_{\mp} = \frac{1}{V} \int d^3 r \int d^3 r' [\phi(t, \mathbf{r},)\ \phi^+(t, \mathbf{r}')]_{\mp}\, e^{-i\,\mathbf{k}\cdot\mathbf{r} + i\,\mathbf{k}'\cdot\mathbf{r}'}$$

$$= \frac{1}{V} \int d^3 r \int d^3 r'\, \delta(\mathbf{r}-\mathbf{r}')\, e^{-i\,\mathbf{k}\cdot\mathbf{r} + i\,\mathbf{k}'\cdot\mathbf{r}'}$$

$$= \frac{1}{V} \int d^3 r\, e^{i\,(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}$$

$$= \delta_{\mathbf{k}\,\mathbf{k}'}$$

$$\mathbf{p} = \int d^3 r\, \phi^+(x) \frac{\hbar}{i} \nabla\, \phi(x)$$

$$= \frac{1}{V} \int d^3 r \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \tilde{a}^+_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}'}(t)\, e^{-i\,\mathbf{k}\cdot\mathbf{r}} \frac{\hbar}{i} \nabla\ e^{i\,\mathbf{k}'\cdot\mathbf{r}}$$

$$= \frac{1}{V} \int d^3 r \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \tilde{a}^+_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}'}(t)\, \hbar\,\mathbf{k}'\ e^{i\,(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}}$$

$$= \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \tilde{a}^+_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}'}(t)\, \hbar\,\mathbf{k}'\ \delta_{\mathbf{k}\,\mathbf{k}'}$$

$$= \sum_{\mathbf{k}} \tilde{a}^+_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}}(t)\, \hbar\,\mathbf{k}$$

Similarly

$$\int d^3 r \frac{\hbar^2}{2m} \nabla \phi^+(x) \cdot \nabla \phi(x) = -\frac{\hbar^2}{2m} \int d^3 x\, \phi^+(x)\, \nabla^2 \phi(x)$$

$$= \sum_{\mathbf{k}} \tilde{a}^+_{\mathbf{k}}(t)\, \tilde{a}_{\mathbf{k}}(t) \frac{\hbar^2 \mathbf{k}^2}{2m}$$

$$\int d^3r\, V(\boldsymbol{x})\, \phi^+(x)\, \phi(x) = \frac{1}{V}\int d^3r \sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}'}(t)\, V(\boldsymbol{r})\, e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}}$$

$$= \frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}'}(t)\, V(\boldsymbol{k}-\boldsymbol{k}') \qquad f(\boldsymbol{k}) = \int d^3r\, f(\boldsymbol{r})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$= \frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}} \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}-\boldsymbol{q}}(t)\, V(\boldsymbol{q}) \qquad \boldsymbol{q} = \boldsymbol{k}-\boldsymbol{k}'$$

$$\frac{1}{2}\int d^3r \int d^3r'\, \phi^+(x)\, \phi^+(x')\, U(\boldsymbol{r}-\boldsymbol{r}')\, \phi(x')\, \phi(x)\Big|_{t=t'}$$

$$= \frac{1}{2V^2}\int d^3r \int d^3r' \sum_{\boldsymbol{k}_1}\sum_{\boldsymbol{k}_2}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} \tilde{a}^+_{\boldsymbol{k}_1}(t)\, \tilde{a}^+_{\boldsymbol{k}_2}(t)\, \tilde{a}_{\boldsymbol{k}_3}(t)\, \tilde{a}_{\boldsymbol{k}_4}(t)$$

$$U(\boldsymbol{r}-\boldsymbol{r}')\, e^{-i(\boldsymbol{k}_1-\boldsymbol{k}_4)\cdot\boldsymbol{r} - i(\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{r}'}$$

$$= \frac{1}{2V^2}\sum_{\boldsymbol{k}_1}\sum_{\boldsymbol{k}_2}\sum_{\boldsymbol{k}_3}\sum_{\boldsymbol{k}_4} \tilde{a}^+_{\boldsymbol{k}_1}(t)\, \tilde{a}^+_{\boldsymbol{k}_2}(t)\, \tilde{a}_{\boldsymbol{k}_3}(t)\, \tilde{a}_{\boldsymbol{k}_4}(t)\, U(\boldsymbol{k}_1-\boldsymbol{k}_4-\boldsymbol{k}_2+\boldsymbol{k}_3)$$

$$= \frac{1}{2V^2}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'} \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}^+_{\boldsymbol{k}'}(t)\, \tilde{a}_{\boldsymbol{k}'-\boldsymbol{q}'}(t)\, \tilde{a}_{\boldsymbol{k}-\boldsymbol{q}}(t)\, U(\boldsymbol{q}-\boldsymbol{q}')$$

where $\boldsymbol{k} = \boldsymbol{k}_1$ $\quad \boldsymbol{q} = \boldsymbol{k}_1 - \boldsymbol{k}_4$

$\boldsymbol{k}' = \boldsymbol{k}_2$ $\quad \boldsymbol{q}' = \boldsymbol{k}_2 - \boldsymbol{k}_3$

$$\rightarrow \quad H = \int d^3r\, \mathcal{H}$$

$$= \sum_{\boldsymbol{k}} \frac{\hbar^2 \boldsymbol{k}^2}{2m}\, \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}}(t)$$

$$+ \frac{1}{V}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}} V(\boldsymbol{q})\, \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}-\boldsymbol{q}}(t)$$

$$+ \frac{1}{2V^2}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q}'} U(\boldsymbol{q}-\boldsymbol{q}')\, \tilde{a}^+_{\boldsymbol{k}}(t)\, \tilde{a}^+_{\boldsymbol{k}'}(t)\, \tilde{a}_{\boldsymbol{k}'-\boldsymbol{q}'}(t)\, \tilde{a}_{\boldsymbol{k}-\boldsymbol{q}}(t)$$

Alternative Lagrangian

$$\mathcal{L} = \frac{1}{2}(i\hbar\varphi^*\partial_t\varphi - i\hbar\varphi\partial_t\varphi^*) - \frac{\hbar^2}{2m}\nabla\varphi^*\cdot\nabla\varphi - V\varphi^*\varphi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi^*)} = -\frac{1}{2}i\hbar\varphi \qquad \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi^*)} = -\frac{1}{2}i\hbar\partial_t\varphi$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla\varphi^*)} = -\frac{\hbar^2}{2m}\nabla\varphi \qquad \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\varphi^*)} = -\frac{\hbar^2}{2m}\nabla^2\varphi$$

$$\frac{\partial\mathcal{L}}{\partial\varphi^*} = \frac{1}{2}i\hbar\partial_t\varphi - V\varphi$$

$$\rightarrow \quad -\frac{\hbar^2}{2m}\nabla^2\varphi - i\hbar\partial_t\varphi + V\varphi = 0$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} = \frac{1}{2}i\hbar\varphi^* \qquad \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} = \frac{1}{2}i\hbar\partial_t\varphi^*$$

$$\frac{\partial \mathcal{L}}{\partial (\nabla \varphi)} = -\frac{\hbar^2}{2m} \nabla \varphi^* \qquad \nabla \cdot \frac{\partial \mathcal{L}}{\partial (\nabla \varphi)} = -\frac{\hbar^2}{2m} \nabla^2 \varphi^*$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{1}{2} i \hbar \partial_t \varphi^* - V \varphi^*$$

$$\rightarrow \qquad -\frac{\hbar^2}{2m} \nabla^2 \varphi^* + i \hbar \partial_t \varphi^* + V \varphi^* = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial (\partial_t \varphi)} = \frac{1}{2} i \hbar \varphi^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial (\partial_t \varphi^*)} = -\frac{1}{2} i \hbar \varphi = (\pi)^*$$

$$\mathcal{H} = \frac{1}{2} (i \hbar \varphi^* \partial_t \varphi - i \hbar \varphi \partial_t \varphi^*) - \mathcal{L} = \frac{\hbar^2}{2m} \nabla \varphi^* \cdot \nabla \varphi + V \varphi^* \varphi$$

2nd Quantization via equal-time commutation relations :

$$[\varphi(\boldsymbol{x}, t), \pi(\boldsymbol{x}', t)]_\mp = i \hbar \delta(\boldsymbol{x} - \boldsymbol{x}') \qquad \text{for} \quad \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix}$$

$$\rightarrow \qquad [\varphi(\boldsymbol{x}, t), \varphi^+(\boldsymbol{x}', t)]_\mp = 2 \delta(\boldsymbol{x} - \boldsymbol{x}')$$