3.3. Real Klein-Gordon Field

Ref: M.Kaku, "Quantum Field Theory", §3.2.

$x^\mu = (c\,t,\ \boldsymbol{r})$ $\qquad$ $x_\mu = \eta_{\mu\nu}\, x^\nu = (c\,t,\ -\boldsymbol{r})$

$\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{c\,\partial t},\ \nabla \right)$ $\qquad$ $\partial^\mu = \dfrac{\partial}{\partial x_\mu} = \left(\dfrac{\partial}{c\,\partial t},\ -\nabla \right)$

$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Caution: Ezawa used $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

Lagrangian

Action: $S = \int d\,t\, L = \int d\,t \int d^3 r\, \mathcal{L} = \dfrac{1}{c} \int d^4 x\, \mathcal{L}$

$$E^2 = \boldsymbol{p}^2 c^2 + m^2 c^4$$

$$E \to i\,\hbar \frac{\partial}{\partial t} \qquad \boldsymbol{p} \to \frac{\hbar}{i} \nabla$$

i.e. $\quad p^\mu = \left(\dfrac{E}{c},\ \boldsymbol{p} \right) \quad \to \quad i\,\hbar\,\partial^\mu = \left(i\,\hbar \dfrac{\partial}{c\,\partial t},\ \dfrac{\hbar}{i} \nabla \right)$

$\quad p_\mu = \left(\dfrac{E}{c},\ -\boldsymbol{p} \right) \quad \to \quad i\,\hbar\,\partial_\mu = \left(i\,\hbar \dfrac{\partial}{c\,\partial t},\ -\dfrac{\hbar}{i} \nabla \right)$

$\to \quad \dfrac{E^2}{c^2} - \boldsymbol{p}^2 = p^\mu\, p_\mu = -\hbar^2\, \partial^\mu\, \partial_\mu = -\hbar^2 \left(\dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \nabla^2 \right) = m^2 c^2$

i.e. $\quad \left(\dfrac{1}{c^2} \dfrac{\partial^2}{\partial t^2} - \nabla^2 + \dfrac{m^2 c^2}{\hbar^2} \right) \phi = 0 \qquad$ (Klein-Gordon eq.)

Set $\quad \mathcal{L} = \dfrac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \dfrac{m^2 c^2}{\hbar^2}\, \phi^2 \right)$

where f is a parameter that may involve m, c, & $\hbar$.

$$\mathcal{L} = \frac{1}{2} f \left[(\partial_\mu \phi)^2 - \frac{m^2 c^2}{\hbar^2}\, \phi^2 \right] = \frac{1}{2} f \left[\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \nabla\phi \cdot \nabla\phi - \frac{m^2 c^2}{\hbar^2}\, \phi^2 \right]$$

$$= \frac{1}{2} f \left(\eta^{\mu\nu}\, \partial_\mu \phi \cdot \partial_\nu \phi - \frac{m^2 c^2}{\hbar^2}\, \phi^2 \right)$$

Kaku's choice: $f = \hbar = c = 1$.

Ezawa's choice: $\quad f = \hbar^2$.

Note: Changing f only changes the units of ϕ.

Units:

$$[\mathcal{L}] = \left[\frac{E}{L^3} \right] = \left[f\, \frac{1}{L^2}\, \phi^2 \right]$$

$$\to \quad [\phi] = \left[\left(\frac{E}{f\,L} \right)^{1/2} \right]$$

$$\frac{\partial\, \mathcal{L}}{\partial\, \phi} = -f\, \frac{m^2 c^2}{\hbar^2}\, \phi$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \frac{1}{2} f\, \eta^{\sigma\nu} (\, \delta^\mu_\sigma \partial_\nu \phi + \delta^\mu_\nu \partial_\sigma \phi)$$

$$= \frac{1}{2} f \,(\, \eta^{\mu\nu} \partial_\nu \phi + \eta^{\sigma\mu} \partial_\sigma \phi)$$

$$= \frac{1}{2} f \,(\, \partial^\mu \phi + \partial^\mu \phi) \;= f \,\partial^\mu \phi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = f \,\partial_\mu \partial^\mu \phi$$

$$\rightarrow \qquad \left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \phi = 0 \qquad \text{Klein-Gordon eq.}$$

$$\text{or} \qquad \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \phi = 0$$

Set $p^\mu \rightarrow i \hbar \partial^\mu$, then the K-G eq becomes

$$\left(-\frac{1}{\hbar^2} p_\mu p^\mu + \frac{m^2 c^2}{\hbar^2} \right) \phi = \frac{1}{\hbar^2} \left(-E^2 c^2 + \mathbf{p}^2 + m^2 c^2 \right) \phi = 0$$

$$\rightarrow \qquad E^2 = \mathbf{p}^2 c^2 + m^2 c^4 \qquad \rightarrow \qquad E = \pm \sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$$

Hamiltonian

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{c\, \partial_0 \phi} = f \frac{1}{c} \partial^0 \phi \;= f \frac{1}{c^2} \dot{\phi}$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

$$= f \frac{1}{c^2} \dot{\phi}^2 - \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 - \nabla\phi \cdot \nabla\phi - \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

$$= \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 + \nabla\phi \cdot \nabla\phi + \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

Energy-Momentum Tensor

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

$$= \frac{1}{2} f \left[2 \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left(\partial_\tau \phi \cdot \partial^\tau \phi - \frac{m^2 c^2}{\hbar^2} \phi^2 \right) \right]$$

$$\therefore \qquad \Theta^{00} = \frac{1}{2} f \left[\frac{2}{c^2} \dot{\phi}^2 - \left(\frac{1}{c^2} \dot{\phi}^2 - \nabla\phi \cdot \nabla\phi - \frac{m^2 c^2}{\hbar^2} \phi^2 \right) \right]$$

$$= \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 + \nabla\phi \cdot \nabla\phi + \frac{m^2 c^2}{\hbar^2} \phi^2 \right) \;= \mathcal{H}$$

$$\Theta^{0j} = \Theta^{j0} = f \frac{1}{c} \dot{\phi} \partial^j \phi \;= c \mathcal{P}^j$$

$$\rightarrow \qquad p^j = \int d^3r\, \mathcal{P}^j = \frac{f}{c^2}\int d^3r\, \dot{\phi}\,\partial^j\phi$$

$$\boldsymbol{p} = -\frac{f}{c^2}\int d^3r\, \dot{\phi}\,\nabla\phi$$

Plane Wave Solutions

Let $\phi_k(x) = e^{-i\,k\cdot x}$ where $k\cdot x = \omega\, t - \boldsymbol{k}\cdot\boldsymbol{r}$

KG eq.: $\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi_k = \left(-k_\mu k^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi_k = 0$

$$\rightarrow \qquad k_\mu k^\mu = k^2 = \left(\frac{\omega}{c}\right)^2 - \boldsymbol{k}^2 = \frac{m^2c^2}{\hbar^2}$$

∴ Plane wave solutions are of the form

$$\phi_{\boldsymbol{k}}^{(\pm)}(x) \propto e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} \mp \omega_{\boldsymbol{k}}\,t)}$$

with $\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2c^2 + \frac{m^2c^4}{\hbar^2}}$

For $\boldsymbol{k}$ continuous,

$$\int d^3r\, e^{i\,(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = (2\pi)^3\,\delta(\boldsymbol{k}-\boldsymbol{k}')$$

$$\int d^3r\, \phi_{\boldsymbol{k}}^{(\pm)*}(x)\,\phi_{\boldsymbol{k}'}^{(\pm)}(x) = \delta(\boldsymbol{k}-\boldsymbol{k}') \rightarrow \qquad \phi_{\boldsymbol{k}}^{(\pm)}(x) = \frac{1}{(2\pi)^{3/2}}\,e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} \mp \omega_{\boldsymbol{k}}\,t)}$$

General Solutions

Plane wave expansion:

$$\phi(x) = \int d^3k\left[A(\boldsymbol{k})\,\phi_{\boldsymbol{k}}^{(+)}(x) + B(\boldsymbol{k})\,\phi_{\boldsymbol{k}}^{(-)}(x)\right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left[A(\boldsymbol{k})\,e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}\,t)} + B(\boldsymbol{k})\,e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} + \omega_{\boldsymbol{k}}\,t)}\right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left[A(\boldsymbol{k})\,e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}\,t)} + B(-\boldsymbol{k})\,e^{-i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}\,t)}\right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left[A(\boldsymbol{k})\,e^{-i\,k\cdot x} + B(-\boldsymbol{k})\,e^{i\,k\cdot x}\right]_{\omega=\omega_{\boldsymbol{k}}}$$

$\mathcal{H}^* = \mathcal{H} \qquad \rightarrow \qquad \phi^*(x) = \phi(x)$ (real field)

$$\phi^*(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[A^*(\boldsymbol{k})\,e^{i\,k\cdot x} + B^*(-\boldsymbol{k})\,e^{-i\,k\cdot x}\right]_{\omega=\omega_{\boldsymbol{k}}}$$

$$\rightarrow \qquad B(-\boldsymbol{k}) = A^*(\boldsymbol{k})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[A(\boldsymbol{k})\,e^{-i\,k\cdot x} + A^*(\boldsymbol{k})\,e^{i\,k\cdot x}\right]_{\omega=\omega_{\boldsymbol{k}}}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}}\left[A(\boldsymbol{k})\,e^{i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}\,t)} + A^*(\boldsymbol{k})\,e^{-i\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}}\,t)}\right]$$

$$\dot{\phi}(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} i\,\omega_{\boldsymbol{k}}\left[-A(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + A^*(\boldsymbol{k})\,e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\right]$$

$$\pi(x) = f\,\frac{1}{c^2}\,\dot{\phi}(x) = f\,\frac{1}{c^2}\int \frac{d^3 k}{(2\pi)^{3/2}} i\,\omega_{\boldsymbol{k}}\left[-A(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)} + A^*(\boldsymbol{k})\,e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_k t)}\right]$$

$$\int d^3 r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\,\phi(x) = \int \frac{d^3 k'}{(2\pi)^{3/2}} \int d^3 r\left[A(\boldsymbol{k})\,e^{i(\boldsymbol{k}'-\boldsymbol{k})\cdot\boldsymbol{r}-i\omega'_k t} + A^*(\boldsymbol{k}')\,e^{-i(\boldsymbol{k}'+\boldsymbol{k})\cdot\boldsymbol{r}+i\omega'_k t}\right]$$

$$= (2\pi)^{3/2}\int d^3 k'\left[A(\boldsymbol{k})\,e^{-i\omega'_k t}\,\delta(\boldsymbol{k}'-\boldsymbol{k}) + A^*(\boldsymbol{k}')\,e^{i\omega'_k t}\,\delta(\boldsymbol{k}'+\boldsymbol{k})\right]$$

$$= (2\pi)^{3/2}\left(A(\boldsymbol{k})\,e^{-i\omega_k t} + A^*(-\boldsymbol{k})\,e^{i\omega_k t}\right)$$

Similarly,

$$\int d^3 r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\dot{\phi}(x) = (2\pi)^{3/2}\; i\,\omega_{\boldsymbol{k}}\left[-A(\boldsymbol{k})\,e^{-i\omega_k t} + A^*(-\boldsymbol{k})\,e^{i\omega_k t}\right]$$

$$\therefore \qquad (2\pi)^{3/2}\,2\,\omega_{\boldsymbol{k}}\,A(\boldsymbol{k})\,e^{-i\omega_k t} = \int d^3 r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left[\omega_{\boldsymbol{k}}\,\phi(x) + i\,\dot{\phi}(x)\right]$$

$$\rightarrow \qquad A(\boldsymbol{k}) = \frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\pi)^{3/2}}\int d^3 r\, e^{i k\cdot x}\left[\omega_{\boldsymbol{k}}\,\phi(x) + i\,\dot{\phi}(x)\right]$$

(Note: Despite the presence of $x = (\boldsymbol{r}, t)$ in the integrand, $A(\boldsymbol{k})$ is independent of t & f.)

$$(2\pi)^{3/2}\,2\,\omega_{\boldsymbol{k}}\,A^*(-\boldsymbol{k})\,e^{i\omega_k t} = \int d^3 r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\left[\omega_{\boldsymbol{k}}\,\phi(x) - i\,\dot{\phi}(x)\right]$$

$$\rightarrow \qquad A^*(-\boldsymbol{k}) = \frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\pi)^{3/2}}\int d^3 r\, e^{-i\omega_k t - i\boldsymbol{k}\cdot\boldsymbol{r}}\left[\omega_{\boldsymbol{k}}\,\phi(x) - i\,\dot{\phi}(x)\right]$$

$$A^*(\boldsymbol{k}) = \frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\pi)^{3/2}}\int d^3 r\, e^{-i k\cdot x}\left[\omega_{\boldsymbol{k}}\,\phi(x) - i\,\dot{\phi}(x)\right]$$

$$= [A(\boldsymbol{k})]^* \qquad \text{since } \phi^* = \phi.$$

Units:

$$[A] = \left[L^3\,\phi\right]$$

$$[\phi] = \left[\left(\frac{E}{f\,L}\right)^{1/2}\right] \rightarrow \qquad [A] = \left[L^3\left(\frac{E}{f\,L}\right)^{1/2}\right] = \left[\left(\frac{E\,L^5}{f}\right)^{1/2}\right]$$

Quantization

Canonical quantization:

ϕ & A become operators. $\quad A^* \to A^+$.

$[\,\phi(\boldsymbol{r}, t), \pi(\boldsymbol{r}', t)\,] = i\,\hbar\,\delta(\boldsymbol{r} - \boldsymbol{r}')$ (equal-time commutator)

$[\,\phi(\boldsymbol{r}, t), \phi(\boldsymbol{r}', t)\,] = [\,\pi(\boldsymbol{r}, t), \pi(\boldsymbol{r}', t)\,] = 0$

$$\rightarrow \qquad \left[\,\phi(\boldsymbol{r}, t), \dot{\phi}(\boldsymbol{r}', t)\,\right] = i\,\frac{\hbar\,c^2}{f}\,\delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$[\,\phi(\boldsymbol{r}, t), \phi(\boldsymbol{r}', t)\,] = \left[\,\dot{\phi}(\boldsymbol{r}, t), \dot{\phi}(\boldsymbol{r}', t)\,\right] = 0$$

Note: For $t = t'$,

$(x - x')^2 = -(\boldsymbol{r} - \boldsymbol{r}')^2 \le 0$ (space-like)

$$A(\boldsymbol{k}) = \frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\pi)^{3/2}}\int d^3 r\, e^{i k\cdot x}\left[\omega_{\boldsymbol{k}}\,\phi(x) + i\,\dot{\phi}(x)\right]$$

$$= \frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\,\pi)^{3/2}} \int d^3 r \left[-i\,c \left(\partial_0 e^{i\,k\cdot x}\right) \phi(x) + i\,e^{i\,k\cdot x}\,c\,\partial_0 \phi(x) \right]$$

$$= i\,\frac{c}{2\,\omega_{\boldsymbol{k}}\,(2\,\pi)^{3/2}} \int d^3 r\;\; e^{i\,k\cdot x}\; \overset{\leftrightarrow}{\partial}_0\; \phi(x)$$

where

$$A\; \overset{\leftrightarrow}{\partial}\; B = A\,\partial B - (\partial A)\,B$$

$$[A] = \left[\frac{L}{T\,T^{-1}}\, L^3\, \frac{1}{L}\, \phi \right] = \left[L^3\, \phi \right] \qquad\qquad \text{(check)}$$

Similarly,

$$A^+(\boldsymbol{k}) = \frac{1}{2\,\omega_{\boldsymbol{k}}(2\,\pi)^{3/2}} \int d^3 r\, e^{-i\,k\cdot x}\left[\omega_{\boldsymbol{k}}\,\phi(x) - i\,\dot{\phi}(x) \right]$$

$$= \frac{c}{2\,\omega_{\boldsymbol{k}}\,(2\,\pi)^{3/2}} \int d^3 r \left[i \left(\partial_0 e^{-i\,k\cdot x}\right) \phi(x) - i\,e^{-i\,k\cdot x}\,\partial_0 \phi(x) \right]$$

$$= -i\,\frac{c}{2\,\omega_{\boldsymbol{k}}(2\,\pi)^{3/2}} \int d^3 r\;\; e^{-i\,k\cdot x}\; \overset{\leftrightarrow}{\partial}_0\; \phi(x)$$

$$= [A(\boldsymbol{k})]^+ \qquad\qquad \text{since}\;\; \phi^+ = \phi$$

$$[A(\boldsymbol{k}),\, A^+(\boldsymbol{k}')] = \left(\frac{1}{2\,\omega_{\boldsymbol{k}}\,(2\,\pi)^{3/2}} \right)^2 \int d^3 r \int d^3 r'\, e^{i\,k\cdot x - i\,k'\cdot x'} \left[\omega_{\boldsymbol{k}}\,\phi(x) + i\,\frac{c^2}{f}\,\pi(x)\,,\; \omega_{\boldsymbol{k}'}\,\phi(x') - i\,\frac{c^2}{f}\,\pi(x') \right]_{t=t'}$$

$$= \left(\frac{1}{2\,\omega_{\boldsymbol{k}}(2\,\pi)^{3/2}} \right)^2 \int d^3 r \int d^3 r'\, e^{i\,k\cdot x - i\,k'\cdot x'}\, \hbar\, \frac{c^2}{f} [\omega_{\boldsymbol{k}}\,\delta(\boldsymbol{r} - \boldsymbol{r}') + \omega_{\boldsymbol{k}'}\,\delta(\boldsymbol{r} - \boldsymbol{r}')\,]_{t=t'}$$

$$= \frac{1}{f} \left(\frac{c}{2\,\omega_{\boldsymbol{k}}(2\,\pi)^{3/2}} \right)^2 \hbar \int d^3 r\;\; e^{i\,(k - k')\cdot x} [\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'}\,]$$

$$= \frac{1}{f} \left(\frac{c}{2\,\omega_{\boldsymbol{k}}\,(2\,\pi)^{3/2}} \right)^2 (2\,\pi)^3\;\; \delta(\boldsymbol{k} - \boldsymbol{k}')\, 2\,\hbar\omega_{\boldsymbol{k}}$$

Similarly, $\qquad [A\,(\boldsymbol{k}),\, A\,(\boldsymbol{k}')\,] = [A^+\,(\boldsymbol{k}),\, A^+(\boldsymbol{k}')\,] = 0$

Setting

$$[a_{\boldsymbol{k}},\, a_{\boldsymbol{k}'}\,] = [a_{\boldsymbol{k}}^+,\, a_{\boldsymbol{k}'}^+\,] = 0$$

$$[a_{\boldsymbol{k}},\, a_{\boldsymbol{k}'}^+] = \delta(\boldsymbol{k} - \boldsymbol{k}')$$

$$\rightarrow \qquad a_{\boldsymbol{k}} = \frac{2\,\omega_{\boldsymbol{k}}}{c} \sqrt{f}\;\; \frac{A(\boldsymbol{k})}{\sqrt{2\,\hbar\omega_{\boldsymbol{k}}}}$$

$$A(\boldsymbol{k}) = \frac{c}{2\,\omega_{\boldsymbol{k}}\,\sqrt{f}} \sqrt{2\,\hbar\omega_{\boldsymbol{k}}}\;\; a_{\boldsymbol{k}} = \frac{\hbar\,c}{\sqrt{f}} \sqrt{\frac{1}{2\,\hbar\,\omega_{\boldsymbol{k}}}}\;\; a_{\boldsymbol{k}}$$

$$\phi(x) = \int \frac{d^3 k}{(2\,\pi)^{3/2}} \left[A(\boldsymbol{k})\, e^{-i\,k\cdot x} + A^+(\boldsymbol{k})\, e^{i\,k\cdot x} \right]$$

$$\therefore \qquad \phi(x) = \frac{\hbar\,c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\,\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}} \left[a_{\boldsymbol{k}}\, e^{-i\,k\cdot x} + a_{\boldsymbol{k}}^+\, e^{i\,k\cdot x} \right]$$

$$\dot{\phi}(x) = \frac{\hbar\,c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\,\pi)^3\, 2\,\hbar\,\omega_{\boldsymbol{k}}}}\, i\,\omega_k \left[-a_{\boldsymbol{k}}\, e^{-i\,k\cdot x} + a_{\boldsymbol{k}}^+\, e^{i\,k\cdot x} \right]$$

$$\pi(x) = f\,\frac{1}{c^2}\,\dot{\phi}(x) = \sqrt{f}\,\frac{\hbar}{c}\int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\hbar\,\omega_k}}\, i\,\omega_k\left[-a_{\mathbf{k}}\, e^{-i k\cdot x} + a_{\mathbf{k}}^{+}\, e^{i k\cdot x}\right]$$

$$A(\mathbf{k}) = i\,\frac{c}{2\,\omega_{\mathbf{k}}\,(2\pi)^{3/2}}\int d^3r\; e^{i k\cdot x}\,\overleftrightarrow{\partial_0}\,\phi(x)$$

$$\rightarrow \qquad a_{\mathbf{k}} = \frac{2\,\omega_{\mathbf{k}}}{c}\sqrt{f}\,\frac{1}{\sqrt{2\hbar\omega_{\mathbf{k}}}}\; i\,\frac{c}{2\,\omega_{\mathbf{k}}(2\pi)^{3/2}}\int d^3r\; e^{i k\cdot x}\,\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \sqrt{f}\int \frac{d^3r}{\sqrt{(2\pi)^3\, 2\hbar\,\omega_{\mathbf{k}}}}\; i\, e^{i k\cdot x}\,\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \sqrt{f}\int \frac{d^3r}{\sqrt{(2\pi)^3\, 2\hbar\,\omega_{\mathbf{k}}}}\;\frac{1}{c}\, e^{i k\cdot x}\left[\omega_{\mathbf{k}}\,\phi(x) + i\,\dot{\phi}(x)\right]$$

$$a_{\mathbf{k}}^{+} = \sqrt{f}\int \frac{d^3r}{\sqrt{(2\pi)^3\, 2\hbar\omega_{\mathbf{k}}}}\,(-i)\, e^{-i k\cdot x}\,\overleftrightarrow{\partial_0}\,\phi(x)$$

$$= \sqrt{f}\int \frac{d^3r}{\sqrt{(2\pi)^3\, 2\hbar\,\omega_{\mathbf{k}}}}\;\frac{1}{c}\, e^{-i k\cdot x}\left[\omega_{\mathbf{k}}\,\phi(x) - i\,\dot{\phi}(x)\right]$$

Eq. of Motion

$$i\hbar\,\partial_t\dot{\phi} = \left[\dot{\phi},\, H\right] = \int d^3r\left[\dot{\phi},\, \mathcal{H}\right]$$

$$= \frac{1}{2} f\int d^3r\left[\dot{\phi}(x),\; \nabla'\phi(x')\cdot\nabla'\phi(x') + \frac{m^2c^2}{\hbar^2}\,\phi(x')^2\right]_{t=t'}$$

$$\left[\dot{\phi}(x),\; \phi(x')^2\right]_{t=t'} = -2i\,\frac{\hbar c^2}{f}\,\delta(\mathbf{r}-\mathbf{r}')\,\phi(x')$$

$$\left[\dot{\phi}(x),\; \nabla'\phi(x')\cdot\nabla'\phi(x')\right]_{t=t'} = \left[\dot{\phi}(x),\; \nabla'\phi(x')\right]\cdot\nabla'\phi(x') + \nabla'\phi(x')\cdot\left[\dot{\phi}(x),\; \nabla'\phi(x')\right]$$

$$= -2i\,\frac{\hbar c^2}{f}\left[\nabla'\delta(\mathbf{r}-\mathbf{r}')\right]\cdot\nabla'\phi(x')$$

$$= -2i\,\frac{\hbar c^2}{f}\left\{\nabla'\cdot\left[\delta(\mathbf{r}-\mathbf{r}')\,\nabla'\phi(x')\right] - \delta(\mathbf{r}-\mathbf{r}')\,\nabla'^2\phi(x')\right\}$$

$$\rightarrow \qquad i\hbar\,\partial_t\dot{\phi} = i\hbar c^2\left[\nabla^2\phi(x) - \frac{m^2c^2}{\hbar^2}\,\phi(x)\right]$$

or $$\frac{\partial^2\phi}{c^2\,\partial t^2} - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\,\phi = 0 \qquad\qquad \text{(K-G eq.)}$$

1-Particle State

Wave function of a particle with momentum $\hbar\mathbf{k}$:

$$\Phi_{\mathbf{k}}(x) = \langle 0 \mid \phi(x) \mid \mathbf{k}\rangle = \langle 0 \mid \phi(x)\, a_{\mathbf{k}}^{+} \mid 0\rangle$$

$$= \frac{\hbar c}{\sqrt{f}}\int \frac{d^3k'}{\sqrt{(2\pi)^3\, 2\hbar\omega_{\mathbf{k}'}}}\left[\langle 0 \mid a_{\mathbf{k}'} \mid \mathbf{k}\rangle\, e^{-i k'\cdot x} + \langle 0 \mid a_{\mathbf{k}'}^{+} \mid \mathbf{k}\rangle\, e^{i k'\cdot x}\right]$$

$$\langle 0 \mid a_{\mathbf{k}'} \mid \mathbf{k} \rangle = \langle \mathbf{k}' \mid \mathbf{k} \rangle = \delta(\mathbf{k} - \mathbf{k}') \qquad \langle 0 \mid a_{\mathbf{k}'}^{+} \mid \mathbf{k} \rangle = 0$$

$$\rightarrow \qquad \Phi_{\mathbf{k}}(x) = \frac{\hbar c}{\sqrt{f}} \frac{1}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}'}}} e^{-i k \cdot x} = \frac{c}{\sqrt{f}} \sqrt{\frac{\hbar}{(2\pi)^3 2\omega_{\mathbf{k}'}}} e^{i \mathbf{k} \cdot \mathbf{r} - i \omega_k t}$$

$$\therefore \qquad \mathbf{p}\,\Phi_{\mathbf{k}}(x) = \frac{\hbar}{i} \nabla \Phi_{\mathbf{k}}(x) = \hbar \mathbf{k}\, \Phi_{\mathbf{k}}(x)$$

$$H\,\Phi_{\mathbf{k}}(x) = i\hbar \frac{\partial}{\partial t} \Phi_{\mathbf{k}}(x) = \hbar\,\omega_{\mathbf{k}}\,\Phi_{\mathbf{k}}(x)$$

$$E_{\mathbf{k}} = \hbar\,\omega_{\mathbf{k}} = \sqrt{\hbar^2 \mathbf{k}^2 c^2 + m^2 c^4}$$

For $m = 0$,

$$E_{\mathbf{k}} = \hbar c \mid \mathbf{k} \mid$$

H & p

$$\mathcal{H} = \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 + \nabla\phi\cdot\nabla\phi + \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

$$H = \int d^3 r\, \mathcal{H} = \int d^3 r \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 + \nabla\phi\cdot\nabla\phi + \frac{m^2 c^2}{\hbar^2} \phi^2 \right)$$

$$\phi(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-i k\cdot x} + a_{\mathbf{k}}^{+} e^{i k\cdot x} \right)$$

$$\dot{\phi}(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\,\omega_{\mathbf{k}}}}\, i\,\omega_k \left(-a_{\mathbf{k}} e^{-i k\cdot x} + a_{\mathbf{k}}^{+} e^{i k\cdot x} \right)$$

$$\rightarrow \qquad \int d^3 r\, \dot{\phi}^2 = \frac{\hbar^2 c^2}{f} \int d^3 r \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}}}} \int \frac{d^3 k'}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}'}}}$$

$$i\,\omega_{\mathbf{k}} \left(-a_{\mathbf{k}} e^{-i k\cdot x} + a_{\mathbf{k}}^{+} e^{i k\cdot x} \right) i\,\omega_{\mathbf{k}'} \left(-a_{\mathbf{k}'} e^{-i k'\cdot x} + a_{\mathbf{k}'}^{+} e^{i k'\cdot x} \right)$$

$$= \frac{\hbar^2 c^2}{f} \int \frac{d^3 k}{2\hbar\,\omega_{\mathbf{k}}} \omega_{\mathbf{k}}^2 \left(-a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2 i \omega_k t} - a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+} e^{2 i \omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{+} a_{\mathbf{k}} \right)$$

$$\therefore \qquad \frac{1}{2} f \int d^3 r \frac{1}{c^2} \dot{\phi}^2 = \frac{1}{4} \int d^3 k\, \hbar\,\omega_{\mathbf{k}} \left(-a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2 i \omega_k t} - a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+} e^{2 i \omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{+} a_{\mathbf{k}} \right)$$

$$\int d^3 r\, \nabla\phi\cdot\nabla\phi = \frac{\hbar^2 c^2}{f} \int d^3 r \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}}}} \int \frac{d^3 k'}{\sqrt{(2\pi)^3 2\hbar\omega_{\mathbf{k}'}}}$$

$$\left(a_{\mathbf{k}} e^{-i k\cdot x} - a_{\mathbf{k}}^{+} e^{i k\cdot x} \right) (i\,\mathbf{k})\cdot(i\,\mathbf{k}') \left(a_{\mathbf{k}'} e^{-i k'\cdot x} - a_{\mathbf{k}'}^{+} e^{i k'\cdot x} \right)$$

$$= \frac{\hbar^2 c^2}{f} \int \frac{d^3 k}{2\hbar\omega_{\mathbf{k}}} \mathbf{k}^2 \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2 i \omega_k t} + a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+} e^{2 i \omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{+} a_{\mathbf{k}} \right)$$

$$= \frac{1}{2f} \int d^3 k \frac{\hbar \mathbf{k}^2 c^2}{\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} e^{-2 i \omega_k t} + a_{\mathbf{k}}^{+} a_{-\mathbf{k}}^{+} e^{2 i \omega_k t} + a_{\mathbf{k}} a_{\mathbf{k}}^{+} + a_{\mathbf{k}}^{+} a_{\mathbf{k}} \right)$$

$$\therefore \qquad \frac{1}{2} f \int d^3 r \; \nabla\phi \cdot \nabla\phi = \frac{1}{4} \int d^3 k \; \frac{\hbar}{\omega_{\boldsymbol{k}}} \boldsymbol{k}^2 c^2 \; \left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$\int d^3 r \; \phi^2 = \frac{\hbar^2 c^2}{f} \int d^3 r \int \frac{d^3 k}{\sqrt{(2\pi)^3 \, 2 \hbar\omega_{\boldsymbol{k}}}} \int \frac{d^3 k'}{\sqrt{(2\pi)^3 \, 2 \hbar\omega_{\boldsymbol{k'}}}}$$

$$\left(a_{\boldsymbol{k}} e^{-i k \cdot x} + a_{\boldsymbol{k}}^+ e^{i k \cdot x}\right)\left(a_{\boldsymbol{k'}} e^{-i k' \cdot x} + a_{\boldsymbol{k'}}^+ e^{i k' \cdot x}\right)$$

$$= \frac{\hbar^2 c^2}{f} \int \frac{d^3 k}{2 \hbar\omega_{\boldsymbol{k}}} \left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$= \frac{1}{2 f} \int d^3 k \; \frac{\hbar c^2}{\omega_{\boldsymbol{k}}} \left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$\frac{1}{2} f \int d^3 r \; \frac{m^2 c^2}{\hbar^2} \phi^2 = \frac{1}{4} \int d^3 k \; \frac{\hbar}{\omega_{\boldsymbol{k}}} \frac{m^2 c^4}{\hbar^2} \left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$\omega_{\boldsymbol{k}} = \sqrt{\boldsymbol{k}^2 c^2 + \frac{m^2 c^4}{\hbar^2}}$$

$$\therefore \qquad H = \frac{1}{4} \int d^3 k \left(\hbar\omega_{\boldsymbol{k}} + \frac{\hbar}{\omega_{\boldsymbol{k}}} \boldsymbol{k}^2 c^2 + \frac{\hbar}{\omega_{\boldsymbol{k}}} \frac{m^2 c^4}{\hbar^2}\right)\left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$+ \frac{1}{4} \int d^3 k \left(-\hbar\omega_{\boldsymbol{k}} + \frac{\hbar}{\omega_{\boldsymbol{k}}} \boldsymbol{k}^2 c^2 + \frac{\hbar}{\omega_{\boldsymbol{k}}} \frac{m^2 c^4}{\hbar^2}\right)\left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t}\right)$$

$$= \frac{1}{2} \int d^3 k \; \hbar\omega_{\boldsymbol{k}} \left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{k'}}^+] = \delta(\boldsymbol{k} - \boldsymbol{k}') \qquad \to \qquad a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ = a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} + \delta(\boldsymbol{0})$$

$$\therefore \qquad H = \int d^3 k \; \hbar\omega_{\boldsymbol{k}} \left[a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} + \frac{1}{2} \delta(\boldsymbol{0})\right]$$

$$= \int d^3 k \; \hbar\omega_{\boldsymbol{k}} \, a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} + \frac{1}{2} \hbar \, \omega_{\boldsymbol{0}}$$

$$\boldsymbol{p} = -\frac{f}{c^2} \int d^3 r \; \dot{\phi} \nabla \phi$$

$$= -\hbar^2 \int d^3 r \int \frac{d^3 k}{\sqrt{(2\pi)^3 \, 2 \hbar \, \omega_{\boldsymbol{k}}}} \int \frac{d^3 k'}{\sqrt{(2\pi)^3 \, 2 \hbar\omega_{\boldsymbol{k'}}}}$$

$$\times \, i \, \omega_k \left(-a_{\boldsymbol{k}} e^{-i k \cdot x} + a_{\boldsymbol{k}}^+ e^{i k \cdot x}\right) i \left(a_{\boldsymbol{k'}} \boldsymbol{k}' e^{-i k' \cdot x} - a_{\boldsymbol{k'}}^+ \boldsymbol{k}' e^{i k' \cdot x}\right)$$

$$= \hbar^2 \int \frac{d^3 k}{2 \hbar \, \omega_{\boldsymbol{k}}} \omega_{\boldsymbol{k}} \boldsymbol{k} \left(a_{\boldsymbol{k}} a_{-\boldsymbol{k}} e^{-2 i \omega_k t} + a_{\boldsymbol{k}}^+ a_{-\boldsymbol{k}}^+ e^{2 i \omega_k t} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$= \frac{1}{2} \int d^3 k \; \hbar \boldsymbol{k} \left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right)$$

$$= \int d^3 k \; \hbar \boldsymbol{k} \left(a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} + \frac{1}{2} \delta(\boldsymbol{0})\right)$$

$$= \int d^3 k \; \hbar \boldsymbol{k} \, a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}$$

Zero Point Energy

$$\langle 0 \mid H \mid 0 \rangle = \int d^3k\ \hbar\omega_{\boldsymbol{k}} \left\langle 0 \mid a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}} + \frac{1}{2}\delta(\mathbf{0}) \;\middle|\; 0 \right\rangle$$

$$= \delta(\mathbf{0}) \int d^3k\ \frac{1}{2}\hbar\omega_{\boldsymbol{k}}$$

$$= \frac{V}{(2\pi)^3} \int d^3k\ \frac{1}{2}\hbar\omega_{\boldsymbol{k}} \qquad \text{(zero-point energy)}$$

Normal ordering:

$$:H: = \int d^3r\ :\mathcal{H}:$$

$$= \frac{1}{2}\int d^3k\ \hbar\,\boldsymbol{k}\ :\left(a_{\boldsymbol{k}} a_{\boldsymbol{k}}^+ + a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}\right):$$

$$= \int d^3k\ \hbar\,\boldsymbol{k}\ a_{\boldsymbol{k}}^+ a_{\boldsymbol{k}}$$

$$= H - \langle 0 \mid H \mid 0 \rangle$$