3.5. Noether Currents

Continuous symmetry (Lie group) → Conserved current / Charge

Ref: H.Kleinert's article at

http://users.physik.fu-berlin.de/~kleinert/b6/psfiles/Chapter-7-conslaw.pdf

Point Mechanics

$$S = \int_{t_a}^{t_b} dt\, L(q, \dot{q}, t)$$

Symmetry variation:

$$q(t) \to q'(t) = f(q, \dot{q})$$

$$\delta_s q(t) \equiv q'(t) - q(t)$$

$$= \epsilon\, \Delta(q, \dot{q}, t) \qquad \text{(infinitesimal case)}$$

Hereafter, $\epsilon \to 0$ is understood so that only continuous symmetries (Lie groups) are considered.

$\delta_s S = 0$

Let A be invariant under the symmetry variation

$$\delta_s S = \int_{t_a}^{t_b} dt\, \delta_s L = 0$$

$$\delta_s L = \frac{\partial L}{\partial q} \delta_s q + \frac{\partial L}{\partial \dot{q}} \delta_s \dot{q} \qquad \delta_s \dot{q} = \frac{d}{dt} \delta_s q$$

$$= \frac{\partial L}{\partial q} \delta_s q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta_s q\right) - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta_s q$$

$$\to \qquad \delta_s S = \int_{t_a}^{t_b} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta_s q + \left.\frac{\partial L}{\partial \dot{q}} \delta_s q\right|_{t_a}^{t_b}$$

$$= \left.\frac{\partial L}{\partial \dot{q}} \epsilon \Delta\right|_{t_a}^{t_b} \qquad \text{for } q(t) \text{ obeying Euler eq.}$$

ϵ, t_b, t_a arbitrary → $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Delta\right) = 0$

i.e., Noether charge $Q = \frac{\partial L}{\partial \dot{q}} \Delta$ is conserved.

$\delta_s L = 0$

Let L be invariant under the symmetry variation

$$0 = \delta_s L = \frac{\partial L}{\partial q} \delta_s q + \frac{\partial L}{\partial \dot{q}} \delta_s \dot{q} \qquad \delta_s \dot{q} = \frac{d}{dt} \delta_s q$$

$$= \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right] \delta_s q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta_s q\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta_s q\right) \qquad \text{for } q(t) \text{ obeying Euler eq.}$$

$$\to \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Delta\right) = 0$$

i.e., Noether charge $Q = \frac{\partial L}{\partial \dot{q}} \Delta$ is conserved.

Euler Eq. Invariant

For symmetry transformations that leaves the Euler eq. unchanged,

$$\delta_s L = \epsilon \frac{d\Lambda(q, \dot{q}, t)}{dt}$$

$$\delta_s S = \epsilon \Lambda \Big|_{t_a}^{t_b} \qquad \forall\ \epsilon, t_b, t_a$$

$$\to \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \Delta - \Lambda\right) = 0$$

i.e., Noether charge $Q = \frac{\partial L}{\partial \dot{q}} \Delta - \Lambda$ is conserved.

Generator

Symmetry variation is generated by the Noether charge:

$$\delta_s q = -\epsilon \{Q, q\}$$

where the Poisson bracket is defined by

$$\{F, G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right)$$

so that

$$\{q_i, p_j\} = \delta_{ij} \qquad \{q_i, q_j\} = \{p_i, p_j\} = 0$$

For our 1-D system

$$\delta_s q = \epsilon \frac{\partial Q}{\partial p} \quad \to \quad \Delta = \frac{\partial Q}{\partial p}$$

$$\therefore \quad Q = \frac{\partial L}{\partial \dot{q}} \Delta - \Lambda = p \frac{\partial Q}{\partial p} - \Lambda$$

Generalization

$$S = \int_{t_a}^{t_b} dt\, L(q, \dot{q}, t) \qquad q(t) = \{q^i(t)\,;\, i = 1, \ldots, n\}$$

Symmetry variation:

$$\delta_s q^i(t) \equiv q'^i(t) - q^i(t)$$

$$= \epsilon^i \Delta_{(i)}(q, \dot{q}, t) \qquad \text{(infinitesimal case)}$$

The parentheses in (i) exempts the index i from the implicit summation over repeated indices.

$$\delta_s L = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right] \delta_s q^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i\right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta_s q^i\right) \qquad \text{for } q(t) \text{ obeying Euler eq.}$$

$$= \epsilon^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\Delta_i\right)$$

$$= \epsilon^i \frac{d}{dt} \Lambda_i$$

→ n Noether charges $Q_i = \frac{\partial L}{\partial \dot{q}^{(i)}} \Delta_i - \Lambda_i$ are conserved.

Symmetry variation:

$$\delta_s q^i(t) \equiv q'^i(t) - q^i(t)$$
$$= \epsilon^{ij} \Delta_j(q, \dot{q}, t) \qquad \text{(infinitesimal case)}$$

$$\delta_s L = \left[\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \right) \right] \delta_s q^i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \right)$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \delta_s q^i \right) \qquad \text{for } q(t) \text{ obeying Euler eq.}$$
$$= \epsilon^{ij} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \Delta_j \right)$$
$$= \epsilon^k \frac{d}{dt} \Lambda_k$$

The 2 sets of variation parameters ϵ^{ij} & ϵ^k must be linearly related, i.e.,

$$\epsilon^{ij} = a_{ijk} \epsilon^k$$

→ $$\delta_s L = \epsilon^k \frac{d}{dt}\left(a_{ijk} \frac{\partial L}{\partial \dot{q}^i} \Delta_j \right)$$

i.e., n Noether charges $Q_k = a_{ijk} \frac{\partial L}{\partial \dot{q}^i} \Delta_j - \Lambda_k$ are conserved.

Applications

Temporal Translation Invariance & Energy Conservation

Let $L = L(q, \dot{q})$ (L is t independent)

Consider $t \to t' = t - \epsilon$

$q \to q'$

with $q(t) = q'(t') = q(t' + \epsilon)$ (for the same orbit)

∴ $q'(t) = q(t + \epsilon)$

→ $\delta_s q(t) = q'(t) - q(t) = q(t + \epsilon) - q(t) \approx \epsilon \dot{q}(t)$ ($\Delta = \dot{q}$)

$\delta_s \dot{q}(t) = \epsilon \ddot{q}(t)$

$$\delta_s L = \frac{\partial L}{\partial q} \delta_s q + \frac{\partial L}{\partial \dot{q}} \delta_s \dot{q} = \epsilon \left(\frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} \right) = \epsilon \frac{dL}{dt} \qquad (\Lambda = L)$$
$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta_s q \right) = \epsilon \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q} \right)$$

→ $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) = 0$$

i.e., $\frac{\partial L}{\partial \dot{q}} \dot{q} - L = H$ is conserved.

The same result can also be obtained from the Noether charge

$$Q = \frac{\partial L}{\partial \dot{q}} \Delta - \Lambda = \frac{\partial L}{\partial \dot{q}} \dot{q} - L = H$$

Q as a generator:

$$\delta_s q = -\epsilon \{Q, q\} \quad \to \quad \dot{q} = -\{H, q\}$$

$$Q = p \frac{\partial Q}{\partial p} - \Lambda \quad \to \quad H = p \frac{\partial H}{\partial p} - L \quad \text{(True for } H = \frac{p^2}{2m} + V\text{)}$$

For the time-dependent variation,

$$\delta_s^t q(t) = \epsilon(t) \dot{q}(t)$$

$$q^\epsilon(t) = q(t) + \delta_s^t q(t) = q(t) + \epsilon(t) \dot{q}(t)$$

$$\dot{q}^\epsilon(t) = \dot{q}(t) + \delta_s^t \dot{q}(t) = \dot{q}(t) + \dot{\epsilon}(t) \dot{q}(t) + \epsilon(t) \ddot{q}(t)$$

$$\delta_\epsilon L^\epsilon = \frac{\partial L^\epsilon}{\partial \epsilon} \epsilon + \frac{\partial L^\epsilon}{\partial \dot{\epsilon}} \dot{\epsilon}$$

$$\frac{\partial L^\epsilon}{\partial \epsilon} = \frac{\partial L^\epsilon}{\partial q^\epsilon} \frac{\partial q^\epsilon}{\partial \epsilon} + \frac{\partial L^\epsilon}{\partial \dot{q}^\epsilon} \frac{\partial \dot{q}^\epsilon}{\partial \epsilon} = \frac{\partial L^\epsilon}{\partial q^\epsilon} \dot{q} + \frac{\partial L^\epsilon}{\partial \dot{q}^\epsilon} \ddot{q}$$

$$\to \quad \left(\frac{\partial L^\epsilon}{\partial \epsilon} \right)_{\epsilon=0} = \frac{\partial L}{\partial q} \dot{q} + \frac{\partial L}{\partial \dot{q}} \ddot{q} = \frac{dL}{dt} = \Lambda$$

$$\frac{\partial L^\epsilon}{\partial \dot{\epsilon}} = \frac{\partial L^\epsilon}{\partial q^\epsilon} \frac{\partial q^\epsilon}{\partial \dot{\epsilon}} + \frac{\partial L^\epsilon}{\partial \dot{q}^\epsilon} \frac{\partial \dot{q}^\epsilon}{\partial \dot{\epsilon}} = \frac{\partial L^\epsilon}{\partial \dot{q}^\epsilon} \dot{q}$$

$$\to \quad \left(\frac{\partial L^\epsilon}{\partial \dot{\epsilon}} \right)_{\epsilon=0} = \frac{\partial L}{\partial \dot{q}} \dot{q}$$

$$\therefore \quad Q = \frac{\partial L^\epsilon}{\partial \dot{\epsilon}} - \Lambda = \frac{\partial L}{\partial \dot{q}} \dot{q} - L$$

$$\delta_s q = \epsilon \frac{\partial Q}{\partial p} = \epsilon \frac{\partial H}{\partial p} = \epsilon \dot{q}$$

Spatial Translation Invariance & Momentum Conservation

L & S invariant under spatial translation.

$$x'^i = x^i + \epsilon^i$$

$$\delta_s x^i(t) = x'^i - x^i = \epsilon^i \qquad (\Delta_i = 1 \quad \forall\, i)$$

$$\delta_s \dot{x}^i(t) = 0$$

$$\delta_s L = \frac{\partial L}{\partial x^i} \delta_s x^i + \frac{\partial L}{\partial \dot{x}^i} \delta_s \dot{x}^i$$

$$= \frac{\partial L}{\partial x^i} \epsilon^i$$

$$= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \epsilon^i$$

$$= 0 \qquad (\Lambda_i = 0 \quad \forall\, i)$$

$$\to \quad \frac{\partial L}{\partial \dot{x}^i} = p_i \qquad \text{is conserved}$$

The same result can also be obtained from the Noether charge

$$Q_i = \frac{\partial L}{\partial \dot{q}^{(i)}} \Delta_i - \Lambda_i = \frac{\partial L}{\partial \dot{x}^j} = p_i$$

Q as a generator:

$$\delta_s q_i = -\epsilon_i \{Q_i, q_i\} \qquad \to \qquad 1 = -\{p_i, x^j\}$$

$$Q_i = p_i \frac{\partial Q_i}{\partial p_i} - \Lambda_i \qquad \to \qquad p_i = p_i \frac{\partial p_i}{\partial p_i} = p_i$$

Rotational Invariance & Angular Momentum Conservation

$$x'^i = R^i{}_j x^j$$

$$R^i{}_j \simeq \delta^i{}_j - \varepsilon_{ijk} \omega^k$$

Note: ε_{ijk} is the antisymmetric symbol.

It takes the same values (± 1 & 0) in all coordinate systems & is therefore not a tensor. Consequently, the placings of the indices in the following calculations have no tensorial significance.

$$\delta_s x^i(t) = x'^i - x^i = -\varepsilon_{ijk} \omega^k x^j = -\omega_{ij} x^j$$

where $\omega_{ij} = \varepsilon_{ijk} \omega^k$ $\qquad$ ($\epsilon_{ij} = -\omega_{ij}$, $a_{ijk} = -\varepsilon_{ijk}$, $\Delta^j = x^j$)

$$\delta_s \dot{x}^i(t) = -\omega_{ij} \dot{x}^j$$

$$\begin{aligned}
\delta_s L &= \frac{\partial L}{\partial x^i} \delta_s x^i + \frac{\partial L}{\partial \dot{x}^i} \delta_s \dot{x}^i \\
&= -\omega_{ij} \left(\frac{\partial L}{\partial x^i} x^j + \frac{\partial L}{\partial \dot{x}^i} \dot{x}^j \right) \\
&= -\omega_{ij} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} \right) x^j + \frac{\partial L}{\partial \dot{x}^i} \dot{x}^j \right] \\
&= -\omega_{ij} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} x^j \right) \\
&= -\frac{1}{2} \omega_{ij} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}^i} x^j - \frac{\partial L}{\partial \dot{x}^j} x^i \right) \\
&= 0 \qquad (\Lambda_{ij} = 0)
\end{aligned}$$

$$\to \qquad L^{ij} = \frac{\partial L}{\partial \dot{x}^j} x^i - \frac{\partial L}{\partial \dot{x}^i} x^j \quad \text{is conserved}$$

Since L^{ij} is antisymmetric, only 3 of them are independent, namely,

$$L_k = \frac{1}{2} \varepsilon_{kij} L^{ij} = (\boldsymbol{x} \times \boldsymbol{p})_k$$

Using the Noether charges, we have

$$Q_k = a_{ijk} \frac{\partial L}{\partial \dot{q}^i} \Delta_j - \Lambda_k = -\varepsilon_{ijk} p_i x_j = (\boldsymbol{x} \times \boldsymbol{p})_k$$

Field Theory

Derivation is analogous to the point mechanics case with

$$L \to \mathcal{L} \qquad \frac{d}{dt} \to \partial_\mu \qquad q(t) \to \phi(x) \qquad x^\mu = (t, \boldsymbol{r})$$

Action: $S = c \int d^4 x \, \mathcal{L}(\phi, \partial\phi, \; x)$

Symmetry transformation:

$$\delta_s \phi(x) = \epsilon \, \Delta(\phi, \partial\phi, \; x)$$

$$\to \qquad \delta_s \mathcal{L} = \epsilon \, \partial_\mu \Lambda^\mu$$

$$\delta_s S = \epsilon \, c \int d^4 x \, \partial_\mu \Lambda^\mu \qquad \text{(surface term)}$$

Conserved Noether current:

$$\partial_\mu j^\mu = 0 \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial \, \partial_\mu \phi} \Delta - \Lambda^\mu$$

$$\partial_\mu j^\mu = \partial_0 j^0 - \nabla \cdot \boldsymbol{j} = 0$$

$$\to \qquad 0 = \int d^3 r \, \partial_\mu j^\mu = \int d^3 r \, \partial_0 j^0 - \oint d\boldsymbol{S} \cdot \boldsymbol{j}$$

$$= \frac{d}{c\,dt} \int d^3 r \, j^0 \qquad \text{for localized } \boldsymbol{j}.$$

i.e. $\frac{d}{dt} Q = 0 \qquad Q = \frac{1}{c} \int d^3 r \, j^0$

Complex Field

Let $\mathcal{L}$ be invariant under the phase (gauge) transformation

$$\phi(x) \;\to\; e^{i\epsilon} \phi(x) \approx (1 + i\,\epsilon)\,\phi(x) \qquad \epsilon = \text{real constant}$$

$$\delta_s \phi(x) = i\,\epsilon\,\phi(x) \qquad (\Delta = i\,\phi)$$

$$\delta_s \phi^+(x) = -i\,\epsilon\,\phi^+(x) \qquad (\Delta^+ = -i\,\phi^+)$$

$$\delta_s \mathcal{L} = 0 \qquad (\Lambda = \Lambda^+ = 0)$$

Conserved (Noether) current :

$$\partial_\mu j^\mu = 0$$

with $j^\mu = i \frac{\partial \mathcal{L}}{\partial \, \partial_\mu \phi} \phi - i \frac{\partial \mathcal{L}}{\partial \, \partial_\mu \phi^+} \phi^+$

$$Q = \frac{1}{c} \int d^3 r \; j^0$$

Schrodinger field:

$$\mathcal{L} = i\,\hbar\,\phi^+ \partial_t \phi - \frac{\hbar^2}{2m} \nabla\phi^+ \cdot \nabla\phi - V\,\phi^+ \phi$$

$$j^0 = -\hbar\,c\,\phi^+ \phi \;= -\hbar\,c\,\rho$$

$$\boldsymbol{j} = -\frac{\hbar^2}{2m} i \left(\phi \nabla\phi^+ - \phi^+ \nabla\phi \right)$$

$$j^\mu = \left(j^0, \; \boldsymbol{j}\right) = -\hbar \left(c\,\rho, \; \frac{\hbar}{2mi} \left(\phi^+ \nabla\phi - \phi \nabla\phi^+ \right) \right)$$

$$= -\hbar \left(c\,\rho, \; \boldsymbol{j}_{\text{pr}} \right)$$

where

$$\boldsymbol{j}_{\text{pr}} = \frac{\hbar}{2\, m\, i} \left(\phi^{+} \nabla \phi - \phi \nabla \phi^{+} \right)$$

is the probability current density.