4.1. Ferromagnets

See Appendix I.

$J > 0$:

$$H_X = -4 \sum_{\langle i,j \rangle} J_{ij} \left[\boldsymbol{S}(i) \cdot \boldsymbol{S}(j) + \frac{1}{4} \rho(i)\, \rho(j) \right]$$

where $\sum_{\langle i,j \rangle} = \frac{1}{2} \sum'_{i,j}$ and $\boldsymbol{S}(i) = \frac{1}{2} \left(c_\uparrow^+(i),\, c_\downarrow^+(i) \right) \boldsymbol{\tau} \begin{pmatrix} c_\uparrow(i) \\ c_\downarrow(i) \end{pmatrix}$, with $\boldsymbol{\tau}$ the Pauli matrices.

Global $O(3)$ symmetry (H_X invariant under same rotation for every $\boldsymbol{S}$).

$$2\, \boldsymbol{S}(i) \cdot \boldsymbol{S}(j) = [\, \boldsymbol{S}(i) + \boldsymbol{S}(j)\,]^2 - \boldsymbol{S}(i)^2 - \boldsymbol{S}(j)^2 = s(s+1) - \frac{3}{2}$$

where $\quad \boldsymbol{s} = \boldsymbol{S}(i) + \boldsymbol{S}(j)$

$\rightarrow \quad \boldsymbol{S}(i) \cdot \boldsymbol{S}(j) = \begin{cases} -\frac{3}{4} & \text{singlet} \\ \frac{1}{4} & \text{triplet} \end{cases}$

$\therefore \quad$ Max $\boldsymbol{S}(i) \cdot \boldsymbol{S}(j) = \frac{1}{4} \quad$ for $\quad \boldsymbol{S}(i) = \boldsymbol{S}(j)$

$\rightarrow \quad$ Ground state = All **S** aligned.
All sites occupied $\rightarrow$ $\rho(i) = 1 \;\; \forall\, i$.

$$\therefore \quad \langle H_X \rangle_g = -4 \sum_{\langle i,j \rangle} J_{ij} \left[\frac{1}{4} + \frac{1}{4} \right]$$

$$= -2 \sum_{\langle i,j \rangle} J_{ij} = -\sum'_{i,j} J_{ij} = -N \sum_{j\,(\neq i)} J_{ij} = -\Delta_x N$$

where $\quad \Delta_x = \sum_{j\,(\neq i)} J_{ij} \quad$ for any fixed i.

Energy associated with electron i is

$$E_i = -4 \sum_{j\,(\neq i)} J_{ij} \left[\boldsymbol{S}(i) \cdot \boldsymbol{S}(j) + \frac{1}{4} \rho(j) \right] \qquad (i \text{ fixed})$$

In ground state with all sites occupied,

$$E_i = -4 \sum_{j\,(\neq i)} J_{ij} \left(\frac{1}{4} + \frac{1}{4} \right) = -2\, \Delta_x$$

$\therefore$ Removing an electron raises the energy by $2\,\Delta_x$.

Energy required to flip spin i is

$$E_f = -4 \sum_{j\,(\neq i)} J_{ij} \left(-2 \times \frac{1}{4} \right) = 2 \sum_{j\,(\neq i)} J_{ij} = 2\, \Delta_x$$

Adding 1 e to filled lattice in ground state:
Added e (at site i) must have opposite spin so

$\boldsymbol{S}(i) \rightarrow 0 \qquad \rho(i) \rightarrow 2$

$$E_i \rightarrow -4 \sum_{j\,(\neq i)} J_{ij} \left(\frac{1}{4} \times 2 \right) = -2\, \Delta_x \text{ (no change)}$$

Thus, no energy is required to add 1 e to filled lattice in GS.

Continuum Approximation

$J_{ij} = J_S$ = spin stiffness

$\boldsymbol{a}^\alpha$ = vector to the α^{th} nearest neighbor.
Inversion symmetry → $\boldsymbol{a}^\alpha$ can be grouped in pairs of opposite directions.
With $\boldsymbol{a}^{-\alpha} \equiv -\boldsymbol{a}^\alpha$, we can set $\boldsymbol{a}^\alpha = \text{sgn}(\alpha)\, \boldsymbol{a}^{|\alpha|}$. Hence

$$\sum_\alpha \boldsymbol{a}^\alpha = \sum_{|\alpha|} \left(\boldsymbol{a}^{|\alpha|} - \boldsymbol{a}^{|\alpha|}\right) = 0$$

$$\sum_{\langle i,j \rangle} J_{ij}\, \boldsymbol{S}(i) \cdot \boldsymbol{S}(j) = \frac{1}{2} \sum_{i,j}{}' J_S\, \boldsymbol{S}(i) \cdot \boldsymbol{S}(j)$$

$$= \frac{1}{2} \sum_{\boldsymbol{x},\alpha} J_S\, \boldsymbol{S}(\boldsymbol{x}) \cdot \boldsymbol{S}\left(\boldsymbol{x} + a^\alpha\right) \qquad \text{(nearest neighbor only)}$$

$$= \frac{1}{2} \sum_{\boldsymbol{x}} J_S \frac{1}{2} \sum_\alpha \boldsymbol{S}(\boldsymbol{x}) \cdot [\boldsymbol{S}(\boldsymbol{x} + a^\alpha) + \boldsymbol{S}(\boldsymbol{x} - a^\alpha)]$$

$$= \frac{1}{2} \sum_{\boldsymbol{x}} J_S \sum_\alpha \boldsymbol{S}(\boldsymbol{x}) \cdot \left[\boldsymbol{S}(\boldsymbol{x}) + \frac{1}{2} a_i^\alpha a_j^\alpha \partial_i \partial_j \boldsymbol{S}(\boldsymbol{x}) + \ldots\right] \qquad (i,\, j = x,\, y,\, z)$$

$$= \frac{1}{2} J_S \sum_{\boldsymbol{x}} \sum_\alpha \left[\boldsymbol{S}(\boldsymbol{x})^2 + \frac{1}{2} a_i^\alpha a_j^\alpha\, \boldsymbol{S}(\boldsymbol{x}) \cdot \partial_i \partial_j \boldsymbol{S}(\boldsymbol{x}) + \ldots\right]$$

$$= \frac{1}{2} J_S \sum_{\boldsymbol{x}} \sum_\alpha \left[\boldsymbol{S}(\boldsymbol{x})^2 + \frac{1}{2} a_i^\alpha a_j^\alpha \partial_i \left[\boldsymbol{S}(\boldsymbol{x}) \cdot \partial_j \boldsymbol{S}(\boldsymbol{x})\right] - \frac{1}{2} a_i^\alpha a_j^\alpha (\partial_i \boldsymbol{S}(\boldsymbol{x})) \cdot \partial_j \boldsymbol{S}(\boldsymbol{x}) + \ldots\right]$$

$$= \frac{1}{2} J_S \sum_{\boldsymbol{x}} \sum_\alpha \left[\boldsymbol{S}(\boldsymbol{x})^2 - \frac{1}{2} a_i^\alpha a_j^\alpha (\partial_i \boldsymbol{S}(\boldsymbol{x})) \cdot \partial_j \boldsymbol{S}(\boldsymbol{x}) + \ldots\right]$$

Total derivative term $\frac{1}{2} a_i^\alpha a_j^\alpha \partial_i \left[\boldsymbol{S}(\boldsymbol{x}) \cdot \partial_j \boldsymbol{S}(x)\right]$ can be dropped from H_X without affecting the dynamics of the system. Setting

$$\boldsymbol{S}(\boldsymbol{x})^2 = \lim_{i \to j} \boldsymbol{S}(i) \cdot \boldsymbol{S}(j) = \frac{1}{4}$$

we have

$$\sum_{i,j}{}' J_S \left[\boldsymbol{S}(\boldsymbol{x})^2 + \frac{1}{4} \rho(i)\, \rho(j)\right] = \sum_{i,j}{}' J_{ij} \left[\frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2} \sum_{i,j}{}' J_S = \frac{1}{2} J_S \sum_\alpha N$$

$$\therefore \qquad H_X = -2 \sum_{i,j}{}' J_{ij} \left[\boldsymbol{S}(i) \cdot \boldsymbol{S}(j) + \frac{1}{4} \rho(i)\, \rho(j)\right]$$

$$\simeq J_S \sum_\alpha \left[\sum_{\boldsymbol{x}} a_i^\alpha a_j^\alpha (\partial_i \boldsymbol{S}(\boldsymbol{x})) \cdot \partial_j \boldsymbol{S}(\boldsymbol{x}) - N\right]$$

Simple Cubic Lattice:
Let $\boldsymbol{a}^\alpha$ be along the Cartesian axes so that

$$a_i^\alpha = \text{sgn}(\alpha)\, a\, \delta_i^{|\alpha|} \qquad (\alpha = \pm 1, \pm 2, \pm 3;\; i = 1, 2, 3)$$

$$\to \qquad \sum_\alpha a_i^\alpha a_j^\alpha = \sum_\alpha a^2 \delta_i^{|\alpha|} \delta_j^{|\alpha|} = 2 a^2 \sum_{|\alpha|} \delta_i^{|\alpha|} \delta_j^{|\alpha|} = 2 a^2 \delta_{ij} \sum_{|\alpha|} \delta_i^{|\alpha|} = 2 a^2 \delta_{ij}$$

$$\sum_{\mathbf{x}} = \frac{1}{a^m} \int d^m x \qquad \text{for } m\text{-D lattice}$$

$\rightarrow$ $$H_X = \frac{2 J_S a^2}{a^m} \int d^m x \, (\partial_i \mathbf{S}(\mathbf{x})) \cdot \partial_i \mathbf{S}(\mathbf{x}) - \Delta_x N \qquad \mathbf{S}^2 = \frac{1}{4}$$

where $$\Delta_x = J_S \sum_{\alpha} = 2 m J_S$$

$$H_X = 2 J_S \left(\int \frac{d^m x}{a^{m-2}} (\partial_i \mathbf{S}(\mathbf{x})) \cdot \partial_i \mathbf{S}(\mathbf{x}) - m N \right)$$

$\rightarrow$ Gapless modes (spin waves).

$\mathbf{S}(\mathbf{x})$ = (nonlinear) sigma field

Spontaneous Broken Symmetry

Symmetry group of H_X is $O(3)$.
Symmetry is spontaneously broken when system is in a specific ground state.
Goldstone theorem → gapless Goldstone mode due to broken global continuous symmetry.

Usage of Nonlinear Sigma Field

Prototype of Bose condensation & spontaneous symmetry breaking.

QH ferromagnets:

Topological excitations in monolayer systems.
Interlayer coherence in bilayer systems.