4.2. Real Klein-Gordon Field

Let

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi + \gamma \phi^2 - \frac{1}{2} g \phi^4 \right) \qquad (g \ge 0)$$

$$= \frac{1}{2} f \left(\frac{1}{c^2} \dot{\phi}^2 - \nabla \phi \cdot \nabla \phi + \gamma \phi^2 - \frac{1}{2} g \phi^4 \right)$$

where

$$\gamma = -\frac{m^2 c^2}{\hbar^2} \qquad \text{for the K-G eq.}$$

To get Ezawa's results, set

$$\partial_\mu \phi \cdot \partial^\mu \phi \to -(\partial_\mu \phi)(\partial^\mu \phi)$$

$$f \to \hbar^2 \qquad \gamma \to \frac{\gamma}{\hbar^2} \qquad g \to \frac{g}{\hbar^2}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = f (\gamma \phi - g \phi^3)$$

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = f \partial^\mu \phi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = f \partial_\mu \partial^\mu \phi$$

→ $\partial_\mu \partial^\mu \phi - \gamma \phi + g \phi^3 = 0$

or $\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi - \gamma \phi + g \phi^3 = 0$

$\mathcal{L}$ is invariant under $\phi(x) \to -\phi(x)$.

$$\begin{cases} \gamma = -\dfrac{m^2 c^2}{\hbar^2} < 0 & \text{real } K-G \text{ field} \\ \gamma > 0 & \text{Bose condensate} \end{cases}$$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{f}{c^2} \dot{\phi}$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{f}{2} \left(\frac{1}{c^2} \dot{\phi}^2 + \nabla \phi \cdot \nabla \phi - \gamma \phi^2 + \frac{1}{2} g \phi^4 \right)$$

For $g = 0$, Fock vacuum is defined by

$$\langle 0 \mid \phi \mid 0 \rangle = 0$$

or $a_{\boldsymbol{k}} \mid 0 \rangle = 0 \quad \to \quad \langle 0 \mid \phi^+ \phi \mid 0 \rangle = 0$

For $g \neq 0$,

$$\langle \underline{0} \mid \phi \mid \underline{0} \rangle = 0$$

with $a_{\boldsymbol{k}} \mid \underline{0} \rangle = 0$

is a false vacuum.

Higgs potential:

$$\mathcal{H}_P = \frac{f}{2} \left(-\gamma \phi^2 + \frac{1}{2} g \phi^4 \right) = \frac{f}{2} \left[\frac{1}{2} g \left(\phi^2 - \frac{\gamma}{g} \right)^2 - \frac{\gamma^2}{2g} \right] \qquad \text{(classical field } \phi\text{)}$$

→ $\dfrac{\partial \mathcal{H}_P}{\partial \phi} = 0 = f\left(-\gamma\,\phi + g\,\phi^3\right)$ $\qquad$ $\dfrac{\partial^2 \mathcal{H}_P}{\partial \phi^2} = f\left(-\gamma + 3\,g\,\phi^2\right)$

∴ $\phi = 0$ or $\phi^2 = \dfrac{\gamma}{g}$

For $\phi = 0$,

$$\frac{\partial^2 \mathcal{H}_P}{\partial \phi^2} = -f\,\gamma \begin{cases} < 0 & \text{for } \gamma > 0 \ \text{(false vacuum)} \\ > 0 & \text{for } \gamma < 0 \ \text{(ground state)} \end{cases}$$

For $\phi^2 = \dfrac{\gamma}{g}$,

Not possible if $\gamma < 0$ since ϕ must be real.

If $\gamma > 0$,

$$\frac{\partial^2 \mathcal{H}_P}{\partial \phi^2} = 2\,f\,\gamma > 0 \qquad \text{(ground state)}$$

For $\gamma > 0$,

$$\mathcal{H}_P = \frac{f}{2}\left[\frac{1}{2}\,g\left(\phi^2 - v^2\right)^2 - \frac{g}{2}\,v^4\right]$$
$$= \frac{f}{2}\left[\frac{1}{2}\,g\left(\phi^2 - v^2\right)^2 - \frac{\gamma^2}{2\,g}\right]$$

Classical vacuum: $\phi = \pm v$.

Symmetry is broken if system is in either ground state.

Fluctuation around ground state $\phi = v$:

→ $\phi(x) = v + \eta(x)$

$\left(\phi^2 - v^2\right)^2 = \left(2\,v\,\eta + \eta^2\right)^2$

$$\mathcal{H} = \frac{f}{2}\left(\frac{1}{c^2}\,\dot{\eta}^2 + \nabla\eta\cdot\nabla\eta + \frac{1}{2}\,g\left(2\,v\,\eta + \eta^2\right)^2 - \frac{\gamma^2}{2\,g}\right)$$

Note: No Goldstone mode since $\mathcal{H} > \mathcal{H}_{GS} = -f\,\dfrac{\gamma^2}{4\,g}$ for any $\eta \neq 0$.

(Symmetry ($\phi \to -\phi$) that's broken is a discrete one.)

$$\mathcal{L} = \frac{1}{2}\,f\left(\partial_\mu\phi\cdot\partial^\mu\phi + \gamma\phi^2 - \frac{1}{2}\,g\,\phi^4\right)$$
$$= \frac{f}{2}\left[\partial_\mu\eta\cdot\partial^\mu\eta - \frac{1}{2}\,g\left(2\,v\,\eta + \eta^2\right)^2\right] \qquad \text{Term } f\,\frac{\gamma^2}{4\,g} \text{ discarded}$$
$$= \frac{f}{2}\left[\frac{1}{c^2}\,\dot{\eta}^2 - \nabla\eta\cdot\nabla\eta - \frac{\gamma}{2\,v^2}\left(2\,v\,\eta + \eta^2\right)^2\right]$$
$$= \frac{f}{2}\left(\frac{1}{c^2}\,\dot{\eta}^2 - \nabla\eta\cdot\nabla\eta - 2\,\gamma\,\eta^2 - 2\,\frac{\gamma}{v}\,\eta^3 - \frac{\gamma}{2\,v^2}\,\eta^4\right)$$

which describes particles with mass $\qquad m_\eta^2 = \dfrac{2\,\gamma\,\hbar^2}{c^2}$ ($\gamma > 0$).

Excited state involving both vacua $\phi = \pm v$ is called a kink, which is a topological soliton.

$$\mathcal{H}_{\text{free}} = \frac{f}{2}\left(\frac{1}{c^2}\,\dot{\eta}^2 + \nabla\eta\cdot\nabla\eta + 2\,\gamma\,\eta^2\right)$$

Quantization:

$$\left[\,\eta(\boldsymbol{r},\,t),\,\dot{\eta}(\boldsymbol{r}',\,t)\,\right] = i\,\frac{\hbar\,c^2}{f}\,\delta(\boldsymbol{r}-\boldsymbol{r}') \qquad \text{(equal-time commutator)}$$

$$[\,\eta(\boldsymbol{r},\,t),\,\eta(\boldsymbol{r}',\,t)\,] = \left[\,\dot{\eta}(\boldsymbol{r},\,t),\,\dot{\eta}(\boldsymbol{r}',\,t)\,\right] = 0$$

Euler eq.

$$\frac{1}{c^2}\,\frac{\partial^2\eta}{\partial t^2} - \nabla^2\eta + 2\,\gamma\eta = 0 \;=\; \frac{1}{c^2}\,\frac{\partial^2\eta}{\partial t^2} - \nabla^2\eta + \frac{m_\eta^2\,c^2}{\hbar^2}\,\eta \qquad \text{(K-G eq.)}$$

Plane wave solutions:

$$\eta_k(x) = e^{-i\,k\cdot x} = e^{-i\,\omega\,t + i\,\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\rightarrow \qquad -\frac{\omega^2}{c^2} + \boldsymbol{k}^2 + \frac{m_\eta^2\,c^2}{\hbar^2} = 0$$

$$\therefore \qquad \omega = \pm\,\omega_{\boldsymbol{k}} \qquad \text{where} \qquad \omega_{\boldsymbol{k}} = c\,\sqrt{\boldsymbol{k}^2 + \frac{m_\eta^2\,c^2}{\hbar^2}}$$

Momentum expansion (see § 3.3):

$$\eta(x) = \frac{\hbar\,c}{\sqrt{f}}\,\int \frac{d^3\,k}{\sqrt{(2\,\pi)^3\,2\,\hbar\omega_{\boldsymbol{k}}}}\,\left(\,\eta_{\boldsymbol{k}}\,e^{-i\,k\cdot x} + \eta_{\boldsymbol{k}}^{+}\,e^{i\,k\cdot x}\,\right)$$

$$H_{\text{free}} = \int d^3\,x\,\mathcal{H}_{\text{free}} \;=\; \int d^3\,k\;\hbar\,\omega_{\boldsymbol{k}}\left(\eta_{\boldsymbol{k}}^{+}\,\eta_{\boldsymbol{k}} + \frac{1}{2}\,\delta(\boldsymbol{0})\right)$$

Ground state $|\,0\,\rangle$ is the Fock vacuum of $\eta_{\boldsymbol{k}}$ with

$$\eta_{\boldsymbol{k}}\;|\,0\,\rangle = 0$$

(False vacuum is $a_{\boldsymbol{k}}\;|\,\underline{0}\,\rangle = 0$)

$$\langle 0\;|\;H_{\text{free}}\;|\;0\rangle = \frac{1}{2}\,\delta\,(\boldsymbol{0})\int d^3\,k\;\hbar\,\omega_{\boldsymbol{k}} \qquad \text{zero-point energy}$$

$$= \frac{1}{2}\,\frac{V}{(2\,\pi)^3}\int d^3\,k\;\hbar\,\omega_{\boldsymbol{k}}$$

$$\phi(x) = \frac{\hbar\,c}{\sqrt{f}}\,\int \frac{d^3\,k}{(2\,\pi)^{3/2}\,\sqrt{2\,\hbar\omega_{\boldsymbol{k}}}}\,\left(\,a_{\boldsymbol{k}}\,e^{-i\,k\cdot x} + a_{\boldsymbol{k}}^{+}\,e^{i\,k\cdot x}\,\right)$$

$$= \phi^{(+)}(x) + \phi^{(-)}(x)$$

where

$$\phi^{(+)}(x) = \frac{\hbar\,c}{\sqrt{f}}\,\int \frac{d^3\,k}{(2\,\pi)^{3/2}\,\sqrt{2\,\hbar\omega_{\boldsymbol{k}}}}\,a_{\boldsymbol{k}}\,e^{-i\,k\cdot x}$$

$$\phi^{(-)}(x) = \frac{\hbar\,c}{\sqrt{f}}\,\int \frac{d^3\,k}{(2\,\pi)^{3/2}\,\sqrt{2\,\hbar\omega_{\boldsymbol{k}}}}\,a_{\boldsymbol{k}}^{+}\,e^{i\,k\cdot x} = \phi^{(+)+}(x)$$

$$\phi(x) = v + \eta(x) \qquad \eta(x) = \eta^{(+)}(x) + \eta^{(-)}(x)$$

$$\rightarrow \qquad \phi^{(\pm)}(x) = \frac{1}{2}\,v + \eta^{(\pm)}(x)$$

with

$$\eta^{(+)}(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}} \eta_{\boldsymbol{k}}\, e^{-i k\cdot x}$$

$$\eta^{(-)}(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}} \eta_{\boldsymbol{k}}^{+}\, e^{i k\cdot x} = \eta^{(+)+}(x)$$

$$\eta_{\boldsymbol{k}} \,|\, 0 \rangle = 0 \qquad \forall\ \boldsymbol{k}$$

$\rightarrow$ $$\eta^{(+)}(x) \,|\, 0 \rangle = 0$$

$\therefore$ $$\phi^{(+)}(x) \,|\, 0 \rangle = \frac{1}{2} v \,|\, 0 \rangle$$

i.e., $|\,0\,\rangle$ is a coherent state of the annihilation operator $\phi^{(+)}$.

$$\int d^3 x\, \phi^{(+)}(x) \,|\, 0 \rangle = V \frac{1}{2} v \,|\, 0 \rangle \qquad \text{where} \qquad V = \int d^3 x$$

$$= \frac{\hbar c}{\sqrt{f}} \int d^3 x \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}} a_{\boldsymbol{k}}\, e^{-i k\cdot x} \,|\, 0 \rangle$$

$$= \frac{\hbar c}{\sqrt{f}} (2\pi)^{3/2} \int \frac{d^3 k}{\sqrt{2\hbar\omega_{\boldsymbol{k}}}} a_{\boldsymbol{k}}\, e^{-i\omega_{\boldsymbol{k}} t}\, \delta(\boldsymbol{k}) \,|\, 0 \rangle \qquad \omega_{\boldsymbol{k}} = c\sqrt{\boldsymbol{k}^2 + \frac{m_\eta^2 c^2}{\hbar^2}}$$

$$= \frac{\hbar c}{\sqrt{f}} \frac{(2\pi)^{3/2}}{\sqrt{2\hbar\omega_0}} a_0\, e^{-i\omega_0 t} \,|\, 0 \rangle \qquad \omega_0 = \frac{m_\eta c^2}{\hbar}$$

$$= \frac{\hbar c}{\sqrt{f}} \frac{(2\pi)^{3/2}}{\sqrt{2 m_\eta}} a_0\, e^{-i\omega_0 t} \,|\, 0 \rangle$$

$\rightarrow$ $$a_0\, e^{-i\omega_0 t} \,|\, 0 \rangle = \frac{\sqrt{f}}{\hbar c} v V \sqrt{\frac{m_\eta}{2(2\pi)^3}} \,|\, 0 \rangle \qquad a_{\boldsymbol{k}} \,|\, 0 \rangle = 0 \qquad \forall\ \boldsymbol{k} \neq 0$$

(Bose condensate)

Ground state $|\,0\,\rangle$ can also be defined by

$$\langle 0 \,|\, \phi(x) \,|\, 0 \rangle = v \neq 0$$

in contrast to the false vacuum

$$\langle \underline{0} \,|\, \phi(x) \,|\, \underline{0} \rangle = 0$$

The static E-L eq. is

$$-\nabla^2 \phi - \gamma\,\phi + g\,\phi^3 = 0$$

or

$$\nabla^2 \eta - 2\gamma\,\eta - 3\frac{\gamma}{v}\eta^2 - \frac{\gamma}{v^2}\eta^3 = 0$$

For $|\,\eta\,| << 1$,

$$\nabla^2 \eta - 2\gamma\,\eta \approx 0$$

$\rightarrow$ $$\langle 0 \,|\, \eta(\boldsymbol{r})\,\eta(\boldsymbol{r}') \,|\, 0 \rangle \rightarrow e^{-|\boldsymbol{r}-\boldsymbol{r}'|/\xi} \qquad \text{for} \qquad |\,\boldsymbol{r} - \boldsymbol{r}'\,| >> \xi$$

where

$$\xi = \frac{1}{\sqrt{2\gamma}} = \frac{\hbar}{m_\eta c} = \text{coherence length}$$