4.4. Sigma Model

Real K-G field:

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi + \gamma \phi^2 - \frac{1}{2} g \phi^4 \right)$$

$$\gamma \phi^2 - \frac{1}{2} g \phi^4 = -\frac{1}{2} g \left(\phi^2 - \frac{\gamma}{g} \right)^2 + \frac{\gamma^2}{2g}$$

$$= -\frac{1}{2} g \left(\phi^2 - v^2 \right)^2 + \frac{1}{2} g v^4 \qquad \left(v = \sqrt{\frac{\gamma}{g}} \right)$$

$$\rightarrow \qquad \mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} g \left(\phi^2 - v^2 \right)^2 + \frac{1}{2} g v^4 \right)$$

Dropping the irrelevant constant term, we have

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} g \left(\phi^2 - v^2 \right)^2 \right)$$

For a set of N real fields, let

$$\boldsymbol{\phi} = (\phi_1, \ \ldots, \ \phi_N)$$

$$\rightarrow \qquad \sum_{a=1}^{N} \phi_a^2 = \boldsymbol{\phi} \cdot \boldsymbol{\phi} = \boldsymbol{\phi}^2$$

Sigma model:

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \frac{1}{2} g \left(\boldsymbol{\phi}^2 - v^2 \right)^2 \right)$$

Note:

$$\mathcal{L} \neq \sum_{a=1}^{N} \mathcal{L}(\phi_a) \quad \text{since} \qquad \sum_{a=1}^{N} \phi_a^4 \neq \boldsymbol{\phi}^4$$

$$\mathcal{H} = \frac{1}{2} f \left(\frac{1}{c^2} \dot{\boldsymbol{\phi}}^2 + \nabla \boldsymbol{\phi} \cdot \nabla \boldsymbol{\phi} + \frac{1}{2} g \left(\boldsymbol{\phi}^2 - v^2 \right)^2 \right)$$

$$\mathcal{H}_P = \frac{1}{4} f g \left(\boldsymbol{\phi}^2 - v^2 \right)^2$$

Let $\quad U^T U = I$

$\quad \boldsymbol{\phi} \rightarrow U \boldsymbol{\phi} \quad$ is an $O(N)$ transformation

Since $\quad \boldsymbol{\phi}^2 = \boldsymbol{\phi}^T \boldsymbol{\phi} \rightarrow \boldsymbol{\phi}^T U^T U \boldsymbol{\phi} = \boldsymbol{\phi}^2$,

$\therefore \quad \mathcal{L} \rightarrow \mathcal{L} \quad$ (invariant)

Let

$$\boldsymbol{\phi} = \boldsymbol{v} + \boldsymbol{\eta}$$

or $\quad \phi_a = v_a + \eta_a \qquad \sum_{a=1}^{N} v_a^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v^2$

Classically, $\mathcal{H}_P$ is positive semi-definite & is minimum when $\boldsymbol{\phi}^2 = v^2$.

Ground state is therefore infinitely degenerate.

A simple choice is

$$v_a = v\, \delta_{aN} \qquad \& \qquad \eta_a \,|\, 0 \rangle = 0 \quad \forall\, a$$

Writing

$\eta_N = \sigma$ & $\eta_a = \pi_a$ $\forall\ a = 1, \ldots, N-1$

we have

$\phi_N = v + \sigma$ & $\phi_a = \pi_a$ $\forall\ a = 1, \ldots, N-1$

and

$\langle 0 \mid \phi_a(x) \mid 0 \rangle = v\, \delta_{aN}$

$\boldsymbol{\phi}^2 = \boldsymbol{\pi}^2 + (v+\sigma)^2$

$\boldsymbol{\phi}^2 - v^2 = \boldsymbol{\pi}^2 + 2\, v\, \sigma + \sigma^2$

$\left(\boldsymbol{\phi}^2 - v^2\right)^2 = 4\, v^2\, \sigma^2 + \ldots$

$\rightarrow \quad \mathcal{L} = \frac{1}{2} f \left(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} + \partial_\mu \sigma \cdot \partial^\mu \sigma - 2\, g\, v^2\, \sigma^2 + \ldots \right)$

$\therefore \quad m_\sigma = \frac{\hbar \sqrt{2\, g}}{c}$ & $m_{\pi_a} = 0\ \forall\ a$

The $\{\pi_a\}$ fields are Goldstone modes associated the spontaneous broken symmetry.

For $N = 3$, the choice $\phi_i = S_i$ and

$\langle 0 \mid \phi_1(x) \mid 0 \rangle = 0$

$\langle 0 \mid \phi_2(x) \mid 0 \rangle = 0$

$\langle 0 \mid \phi_3(x) \mid 0 \rangle = v$

corresponds to spontaneous magnetization along the z-axis.

Consider the case $g \rightarrow \infty$ while v remains finite.
$\mathcal{L}$ can be finite only if $\mathcal{H}_P = 0$, i.e.,

$\boldsymbol{\phi}^2 = v^2$

so that

$\mathcal{L} = \frac{1}{2} f\, \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi}$

Ground state is infinitely degenerate.

Constraint $\boldsymbol{\phi}^2 = v^2$ means only $N-1$ field are independent.
Choosing $\boldsymbol{\pi}$ as the independent fields,

$\phi_N = \sqrt{v^2 - \boldsymbol{\pi}^2}$ (massive σ boson eliminated)

Since $\boldsymbol{\pi}$ transform nonlinearly under the $O(N)$ transformation, this is called non-linear sigma model.