4.5. Schrodinger Field

Hamiltonian

Classical field

$$\mathcal{H}_P = \frac{1}{2} g\left(\phi^* \phi - v^2\right)^2$$

$$\mathcal{L} = i\hbar\phi^* \partial_t \phi - \frac{\hbar^2}{2m}\nabla\phi^* \cdot \nabla\phi - \frac{1}{2} g\left(\phi^* \phi - v^2\right)^2$$

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = i\hbar\phi^* \qquad\qquad \pi^* = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi^*)} = 0 \neq (\pi)^*$$

$$\mathcal{H} = \frac{\hbar^2}{2m}\nabla\phi^* \cdot \nabla\phi + \frac{1}{2} g\left(\phi^* \phi - v^2\right)^2$$

$\mathcal{H}$ is invariant under gauge transformation

$$\phi(x) \to e^{i\alpha}\phi(x) \qquad \alpha = \text{real constant}$$

Minimum of $\mathcal{H}$ is at

$$\phi^* \phi = v^2$$

or $\quad \phi = v\, e^{i\beta} \qquad \beta = \text{real constant}$

→ Ground state (of quantized field) is infinitely degenerated.

Gauge symmetry is spontaneously broken when system assumes a ground state with a particular value of β.

Since

$$\rho = \phi^* \phi = \text{particle density}$$

chemical potential of system is

$$\mu \equiv \frac{\partial\mathcal{H}}{\partial\rho} = g\left(\rho - v^2\right)$$

→ $\mu = 0$ for the ground state.

For excited states, set

$$\phi(x) = v + \eta(x) \qquad (\beta = 0 \text{ chosen})$$

→ $$\phi^* \phi = v^2 + v(\eta + \eta^*) + \eta^* \eta$$

$$\mathcal{H}_P = \frac{1}{2} g\left[v(\eta + \eta^*) + \eta^* \eta\right]^2$$

$$= \frac{1}{2} g v^2\left(\eta^2 + 2\eta^* \eta + \eta^{*2}\right) + \mathcal{H}_{\text{int}}$$

where

$$\mathcal{H}_{\text{int}} = \frac{1}{2} g\left[2v(\eta + \eta^*) + \eta^* \eta\right]\eta^* \eta$$

Quantization :

$$[\phi(t,\boldsymbol{r}), \pi(t,\boldsymbol{r}')] = i\hbar\delta(\boldsymbol{r} - \boldsymbol{r}')$$

→ $$[\phi(t,\boldsymbol{r}), \phi^+(t,\boldsymbol{r}')] = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$[\eta(t,\boldsymbol{r}), \eta^+(t,\boldsymbol{r}')] = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$[\phi(t,\boldsymbol{r}), \phi(t,\boldsymbol{r}')] = [\phi^+(t,\boldsymbol{r}), \phi^+(t,\boldsymbol{r}')] = 0$$

→ $$[\eta(t,\boldsymbol{r}), \eta(t,\boldsymbol{r}')] = [\eta^+(t,\boldsymbol{r}), \eta^+(t,\boldsymbol{r}')] = 0$$

$$\mathcal{H} = \frac{\hbar^2}{2m} \nabla\eta^+ \cdot \nabla\eta + \frac{1}{2} g v^2 \left(\eta^2 + \eta\eta^+ + \eta^+ \eta + \eta^{+2}\right) + \mathcal{H}_{\text{int}}$$

$$\mathcal{H}_{\text{int}} = \frac{1}{2} g v \left[\left(\eta + \eta^+\right) \eta^+ \eta + \eta^+ \eta \left(\eta + \eta^+\right) + \frac{1}{v} \left(\eta^+ \eta\right)^2 \right]$$

Plane Wave Expansion

$$\eta(x) = \int \frac{d^3 k}{(2\pi)^{3/2}} \eta_{\boldsymbol{k}} e^{-i k x}$$

where

$$k x = \omega_{\boldsymbol{k}} t - \boldsymbol{k} \cdot \boldsymbol{r} \qquad \omega_{\boldsymbol{k}} = \frac{\epsilon_{\boldsymbol{k}}}{\hbar} = \frac{\hbar \boldsymbol{k}^2}{2m}$$

$$[\eta(t, \boldsymbol{r}), \eta^+(t, \boldsymbol{r}')] = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\rightarrow \qquad [\eta_{\boldsymbol{k}}, \eta_{\boldsymbol{l}}^+] = \delta(\boldsymbol{k} - \boldsymbol{l})$$

$$[\eta(t, \boldsymbol{r}), \eta(t, \boldsymbol{r}')] = [\eta^+(t, \boldsymbol{r}), \eta^+(t, \boldsymbol{r}')] = 0$$

$$\rightarrow \qquad [\eta_{\boldsymbol{k}}, \eta_{\boldsymbol{l}}] = [\eta_{\boldsymbol{k}}^+, \eta_{\boldsymbol{l}}^+] = 0$$

Fock vacuum of η :

$$\eta_{\boldsymbol{k}} \mid 0 \rangle = 0 \qquad \forall \ \boldsymbol{k}$$

$$\langle 0 \mid \eta_{\boldsymbol{k}}^+ = 0$$

$$\rightarrow \qquad \langle 0 \mid \eta(x) \mid 0 \rangle = \int \frac{d^3 k}{(2\pi)^{3/2}} \langle 0 \mid \eta_{\boldsymbol{k}} \mid 0 \rangle e^{-i k x} = 0$$

$$\begin{aligned} \langle 0 \mid \phi(x) \mid 0 \rangle &= \langle 0 \mid v + \eta(x) \mid 0 \rangle = v \\ &= \langle 0 \mid v \mid 0 \rangle + \int \frac{d^3 k}{(2\pi)^{3/2}} \langle 0 \mid \eta_{\boldsymbol{k}} \mid 0 \rangle e^{-i k x} \\ &= v \end{aligned}$$

$$\begin{aligned} \langle 0 \mid \rho(x) \mid 0 \rangle &= \langle 0 \mid \phi^+(x) \phi(x) \mid 0 \rangle \\ &= \langle 0 \mid [v + \eta^+(x)][v + \eta(x)] \mid 0 \rangle \\ &= v^2 \end{aligned}$$

$$\mathcal{H}_{\text{free}} = \frac{\hbar^2}{2m} \nabla\eta^+ \cdot \nabla\eta + \frac{1}{2} g v^2 \left(\eta^2 + \eta\eta^+ + \eta^+ \eta + \eta^{+2}\right)$$

$$\begin{aligned} H_{\text{free}} &= \int d^3 r \, \mathcal{H}_{\text{free}} \\ &= \int d^3 k \, \mathcal{H}_{\text{free}}(\boldsymbol{k}) \end{aligned}$$

Applying the plane wave expansion rules (Fourier transform with factor $(2\pi)^3$ absorbed in the plane wave normalization, see §3.2),

$$f^2 \rightarrow f_{\boldsymbol{k}} f_{-\boldsymbol{k}} \qquad f^{+2} \rightarrow f_{\boldsymbol{k}}^+ f_{-\boldsymbol{k}}^+$$

$$f^+ f \rightarrow \frac{1}{2} \left(f_{\boldsymbol{k}}^+ f_{\boldsymbol{k}} + f_k f_k^+\right) = f_{\boldsymbol{k}}^+ f_{\boldsymbol{k}} + \frac{1}{2} \delta(0)$$

we have

$$\mathcal{H}_{\text{free}}(\boldsymbol{k}) = \frac{1}{2} \varepsilon_{\boldsymbol{k}} \left(\eta_{\boldsymbol{k}}^+ \eta_{\boldsymbol{k}} + \eta_{\boldsymbol{k}} \eta_{\boldsymbol{k}}^+\right) + \frac{1}{2} g v^2 \left(\eta_{\boldsymbol{k}} \eta_{-\boldsymbol{k}} + \eta_{\boldsymbol{k}} \eta_{\boldsymbol{k}}^+ + \eta_{\boldsymbol{k}}^+ \eta_{\boldsymbol{k}} + \eta_{\boldsymbol{k}}^+ \eta_{-\boldsymbol{k}}^+\right)$$

$$= \varepsilon_k\, \eta_k^+ \eta_k + U_k \left(\eta_k\, \eta_{-k} + 2\, \eta_k^+ \eta_k + \eta_k^+ \eta_{-k}^+\right) + \left(\frac{1}{2}\,\varepsilon_k + U_k\right) \delta(0)$$

$$= (\varepsilon_k + 2\, U_k)\, \eta_k^+ \eta_k + U_k \left(\eta_k\, \eta_{-k} + \eta_k^+ \eta_{-k}^+\right) + \left(\frac{1}{2}\,\varepsilon_k + U_k\right) \delta(0)$$

where $\qquad U_k = \frac{1}{2}\, g\, v^2$

∴ Fock vacuum $|\,0\rangle$ is not the ground state if $v \neq 0$.

Ground State

Eigenstates of $\mathcal{H}_{\text{free}}(\boldsymbol{k})$ are found by diagonalization by means of a Bogoliubov transformation (see BogoliubovTransformation.pdf):

$$\zeta_k = c_k\, \eta_k + s_k\, \eta_{-k}^+ \qquad\qquad \zeta_k^+ = c_k\, \eta_k^+ + s_k\, \eta_{-k}$$

where

$$c_k = \cosh \tau_k \qquad\qquad s_k = \sinh \tau_k$$
$$c_k^2 - s_k^2 = 1$$

with

$$[\,\tau_k,\, \tau_l^+] = \delta(\boldsymbol{k} - \boldsymbol{l})$$
$$[\,\tau_k,\, \tau_l] = [\,\tau_k^+,\, \tau_l^+] = 0$$

Inverse transform is

$$\eta_k = c_k\, \zeta_k - s_k\, \zeta_{-k}^+ \qquad\qquad \eta_k^+ = c_k\, \zeta_k^+ - s_k\, \zeta_{-k}$$

Hence,

$$\eta_k^+ \eta_k = c_k^2\, \zeta_k^+ \zeta_k + s_k^2\, \zeta_{-k}\, \zeta_{-k}^+ - c_k\, s_k\left(\zeta_k^+ \zeta_{-k}^+ + \zeta_{-k}\, \zeta_k\right)$$
$$\eta_k\, \eta_{-k} = c_k^2\, \zeta_k\, \zeta_{-k} + s_k^2\, \zeta_{-k}^+ \zeta_k^+ - c_k\, s_k\left(\zeta_k\, \zeta_k^+ + \zeta_{-k}^+ \zeta_{-k}\right)$$
$$\eta_k^+ \eta_{-k}^+ = c_k^2\, \zeta_k^+ \zeta_{-k}^+ + s_k^2\, \zeta_{-k}\, \zeta_k - c_k\, s_k\left(\zeta_k^+ \zeta_k + \zeta_{-k}\, \zeta_{-k}^+\right)$$

Since $\mathcal{H}_{\text{free}}$ is invariant under $\boldsymbol{k} \to -\boldsymbol{k}$, each term in $\mathcal{H}_{\text{free}}(\boldsymbol{k})$ can be replaced by one obtained by taking $\boldsymbol{k} \to -\boldsymbol{k}$. For example,

$$s_k^2\, \zeta_{-k}\, \zeta_{-k}^+ = s_{-k}^2\, \zeta_k\, \zeta_k^+ = s_k^2\, \zeta_k\, \zeta_k^+$$

where we've assumed $\tau_{-k} = \tau_k$ so that

$$c_{-k} = c_k \qquad\qquad s_{-k} = s_k$$

$$\to \qquad \mathcal{H}_{\text{free}}(\boldsymbol{k}) = (\varepsilon_k + 2\, U_k)\left[\, c_k^2\, \zeta_k^+ \zeta_k + s_k^2\, \zeta_k\, \zeta_k^+ - c_k\, s_k\left(\zeta_k^+ \zeta_{-k}^+ + \zeta_{-k}\, \zeta_k\right)\right]$$
$$+ U_k \left(c_k^2 + s_k^2\right)\left(\zeta_k\, \zeta_{-k} + \zeta_{-k}^+ \zeta_k^+\right)$$
$$-2\, U_k\, c_k\, s_k\left(\zeta_k\, \zeta_k^+ + \zeta_k^+ \zeta_k\right) + \left(\frac{1}{2}\,\varepsilon_k + U_k\right) \delta(0)$$

$$= (\varepsilon_k + 2\, U_k)\left[\,\left(c_k^2 + s_k^2\right) \zeta_k^+ \zeta_k + s_k^2\, \delta(0)\,\right]$$
$$+ \left[-(\varepsilon_k + 2\, U_k)\, c_k\, s_k + U_k\left(c_k^2 + s_k^2\right)\right]\left(\zeta_k\, \zeta_{-k} + \zeta_{-k}^+ \zeta_k^+\right)$$
$$-2\, U_k\, c_k\, s_k\,[\,2\, \zeta_k^+ \zeta_k + \delta(0)\,]$$
$$+ \left(\frac{1}{2}\,\varepsilon_k + U_k\right) \delta(0)$$

$$= \left[(\varepsilon_k + 2\, U_k)\left(c_k^2 + s_k^2\right) - 4\, U_k\, c_k\, s_k\,\right] \zeta_k^+ \zeta_k$$
$$+ \left[-(\varepsilon_k + 2\, U_k)\, c_k\, s_k + U_k\left(c_k^2 + s_k^2\right)\right]\left(\zeta_k\, \zeta_{-k} + \zeta_{-k}^+ \zeta_k^+\right)$$
$$+ \left[(\varepsilon_k + 2\, U_k)\, s_k^2 - 2\, U_k\, c_k\, s_k + \frac{1}{2}\,\varepsilon_k + U_k\right] \delta(0)$$

$\mathcal{H}_{\text{free}}(\boldsymbol{k})$ is diagonal if

$$-(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} + U_{\boldsymbol{k}}\left(c_{\boldsymbol{k}}^2 + s_{\boldsymbol{k}}^2\right) = 0$$

so that

$$\mathcal{H}_{\text{free}}(\boldsymbol{k}) = \left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\left(c_{\boldsymbol{k}}^2 + s_{\boldsymbol{k}}^2\right) - 4\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}}\right]\zeta_{\boldsymbol{k}}^+\,\zeta_{\boldsymbol{k}} + \left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,s_{\boldsymbol{k}}^2 - 2\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} + \frac{1}{2}\,\varepsilon_{\boldsymbol{k}} + U_{\boldsymbol{k}}\right]\delta(0)$$

Dropping the subscript $\boldsymbol{k}$ for clarity, we have

$$(\varepsilon + 2\,U)^2\,c^2\,s^2 = U^2\left(c^2 + s^2\right)^2$$

$$(\varepsilon + 2\,U)^2\,c^2\left(c^2 - 1\right) = U^2\left(2\,c^2 - 1\right)^2$$

$$\to \quad A\,c^2\left(c^2 - 1\right) = \left(2\,c^2 - 1\right)^2 \qquad \text{where} \qquad A = \left(\frac{\varepsilon + 2\,U}{U}\right)^2$$

$$(A - 4)\,c^4 - (A - 4)\,c^2 - 1 = 0$$

$$\to \quad c^2 = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{4}{A - 4}}\right) = \frac{1}{2}\left(1 \pm \sqrt{\frac{A}{A - 4}}\right)$$

$$A - 4 = \frac{(\varepsilon + 2\,U)^2 - 4\,U^2}{U^2} = \frac{\varepsilon\,(\varepsilon + 4\,U)}{U^2}$$

$$\frac{A}{A - 4} = \frac{(\varepsilon + 2\,U)^2}{\varepsilon\,(\varepsilon + 4\,U)}$$

$$\to \quad c^2 = \frac{1}{2}\left(1 \pm \frac{\varepsilon + 2\,U}{\sqrt{\varepsilon\,(\varepsilon + 4\,U)}}\right)$$

Since $c_{\boldsymbol{k}}$ is real $\forall\ \boldsymbol{k}$, we set

$$c_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{\sqrt{\varepsilon_{\boldsymbol{k}}\,(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})}}\right) = \frac{1}{2}\left(1 + \frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)$$

where

$$E_{\boldsymbol{k}} = \sqrt{\varepsilon_{\boldsymbol{k}}\,(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})}$$

$$s_k^2 = c_{\boldsymbol{k}}^2 - 1 = \frac{1}{2}\left(\frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} - 1\right)$$

$$c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} = \frac{1}{2}\sqrt{\left(\frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)^2 - 1}$$

$$= \frac{1}{2\,E_{\boldsymbol{k}}}\sqrt{(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})^2 - \varepsilon_{\boldsymbol{k}}\,(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})} = \frac{U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}$$

$$(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\left(c_{\boldsymbol{k}}^2 + s_{\boldsymbol{k}}^2\right) - 4\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} = (\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,\frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} - 4\,\frac{U_{\boldsymbol{k}}^2}{E_{\boldsymbol{k}}}$$

$$= \frac{1}{E_{\boldsymbol{k}}}\left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})^2 - 4\,U_{\boldsymbol{k}}^2\right]$$

$$= \frac{1}{E_{\boldsymbol{k}}}\,\varepsilon_{\boldsymbol{k}}(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}}) = E_{\boldsymbol{k}}$$

$$(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,s_{\boldsymbol{k}}^2 - 2\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} + \frac{1}{2}\,\varepsilon_{\boldsymbol{k}} + U_{\boldsymbol{k}}$$

$$= (\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,\frac{1}{2}\left(\frac{\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} - 1\right) - 2\,\frac{U_{\boldsymbol{k}}^2}{E_{\boldsymbol{k}}} + \frac{1}{2}\,\varepsilon_{\boldsymbol{k}} + U_{\boldsymbol{k}}$$

$$= \frac{(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})^2 - 4\,U_{\boldsymbol{k}}^2}{2\,E_{\boldsymbol{k}}}$$

$$= \frac{1}{2}\,E_{\boldsymbol{k}}$$

$$\therefore \qquad \mathcal{H}_{\text{free}}(\boldsymbol{k}) = \left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,(c_{\boldsymbol{k}}^2 + s_{\boldsymbol{k}}^2) - 4\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}}\right]\zeta_{\boldsymbol{k}}^+\,\zeta_{\boldsymbol{k}}$$

$$+\left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,s_{\boldsymbol{k}}^2 - 2\,U_{\boldsymbol{k}}\,c_{\boldsymbol{k}}\,s_{\boldsymbol{k}} + \frac{1}{2}\,\varepsilon_{\boldsymbol{k}} + U_{\boldsymbol{k}}\right]\delta(0)$$

$$= E_{\boldsymbol{k}}\left[\zeta_{\boldsymbol{k}}^+\,\zeta_{\boldsymbol{k}} + \frac{1}{2}\,\delta(0)\right]$$

$$H_{\text{free}} = \int d^3 k\,\mathcal{H}_{\text{free}}(\boldsymbol{k}) = \int d^3 k\,E_{\boldsymbol{k}}\left[\zeta_{\boldsymbol{k}}^+\,\zeta_{\boldsymbol{k}} + \frac{1}{2}\,\delta(0)\right]$$

Ground state is the Fock vacuum of $\zeta_{\boldsymbol{k}}$:

$$\zeta_{\boldsymbol{k}}\,|\,0\rangle\rangle = 0$$

Superfluid Mode

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2\,\boldsymbol{k}^2}{2\,m} \qquad U_{\boldsymbol{k}} = \frac{1}{2}\,g\,v^2$$

$$\rightarrow \qquad E_{\boldsymbol{k}} = \sqrt{\varepsilon_{\boldsymbol{k}}\,(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})} = \sqrt{\frac{\hbar^2\,\boldsymbol{k}^2}{2\,m}\left(\frac{\hbar^2\,\boldsymbol{k}^2}{2\,m} + 2\,g\,v^2\right)}$$

$$= \hbar\,|\boldsymbol{k}|\,\sqrt{\frac{\hbar^2\,\boldsymbol{k}^2}{4\,m^2} + \frac{g\,v^2}{m}}$$

$$\simeq \hbar\,|\boldsymbol{k}|\,v\,\sqrt{\frac{g}{m}} \quad \text{for small } |\boldsymbol{k}|$$

$$\propto |\boldsymbol{k}| \qquad \left(\text{superfluid mode with velocity } v\,\sqrt{\frac{g}{m}}\right)$$

Vacua

$$\eta_{\boldsymbol{k}}\,|\,0\rangle = 0 \qquad \zeta_{\boldsymbol{k}}\,|\,0\rangle\rangle = 0$$

Let

$$\zeta_{\boldsymbol{k}} = e^{-i\,G}\,\eta_{\boldsymbol{k}}\,e^{i\,G}$$

with $G^+ = G \quad \rightarrow \quad \zeta_{\boldsymbol{k}}^+ = e^{-i\,G}\,\eta_{\boldsymbol{k}}^+\,e^{i\,G}$

$$\zeta_{\boldsymbol{k}}\,|\,0\rangle\rangle = 0$$

$$\rightarrow \qquad e^{-i\,G}\,\eta_{\boldsymbol{k}}\,e^{i\,G}\,|\,0\rangle\rangle = 0$$

$$\eta_{\boldsymbol{k}}\,e^{i\,G}\,|\,0\rangle\rangle = 0$$

$$\rightarrow \qquad e^{i\,G}\,|\,0\rangle\rangle \propto |\,0\rangle$$

$$\langle 0\,|\,0\rangle = \langle\langle 0\,|\,0\rangle\rangle = 1$$

→ $e^{iG}\,|\,0\rangle\rangle = |\,0\rangle$

$|\,0\rangle\rangle = e^{-iG}\,|\,0\rangle$

From BogoliubovTransformation.pdf, we have

$$G = \frac{1}{2}\, i \int d^3 k\, \theta_{\mathbf{k}} \left(\eta_{\mathbf{k}}\, \eta_{-\mathbf{k}} - \eta^+_{\mathbf{k}}\, \eta^+_{-\mathbf{k}}\right)$$

Particle & Phase Version

Let (c.f. §1.7):

$\phi(x) = e^{i\,\Theta(x)} \sqrt{\rho(x)}$ → $\phi^+(x) = \sqrt{\rho(x)}\; e^{-i\,\Theta(x)}$

where $\rho^+(x) = \rho(x)$ & $\Theta^+(x) = \Theta(x)$

∴ $\phi^+(x)\,\phi(x) = \rho(x)$ $\phi(x)\,\phi^+(x) = e^{i\,\Theta(x)}\,\rho(x)\,e^{-i\,\Theta(x)}$

$[\,\phi(t, \mathbf{r}),\, \phi^+(t, \mathbf{r}')\,] = \delta(\mathbf{r} - \mathbf{r}')$

→ $[\,\phi(x),\, \phi^+(x)\,] = \delta(\mathbf{0})$

$= e^{i\,\Theta(x)}\,\rho(x)\,e^{-i\,\Theta(x)} - \rho(x)$

$= \left[\, e^{i\,\Theta(x)},\, \rho(x)\,\right] e^{-i\,\Theta(x)}$

∴ $\left[\, e^{i\,\Theta(x)},\, \rho(x)\,\right] = e^{i\,\Theta(x)}\,\delta(\mathbf{0})$

$\left[\, e^{i\,\Theta(x)},\, \rho(x')\,\right]_{t=t'} = e^{i\,\Theta(x)}\,\delta(\mathbf{r} - \mathbf{r}')$

To lowest order in Θ on both sides,

$[\, i\,\Theta(x),\, \rho(x')\,]_{t=t'} = \delta(\mathbf{r} - \mathbf{r}')$

$[\,\Theta(x),\, \rho(x')\,]_{t=t'} = -i\,\delta(\mathbf{r} - \mathbf{r}')$

Let $\rho(x) = v^2\,[\,1 + \sigma(x)\,]$ $\theta(x) = 2\,\Theta(x)$

→ $\phi^+(x)\,\phi(x) = v^2\,[\,1 + \sigma(x)\,]$

$\phi(x) = e^{i\,\theta(x)/2}\, v \sqrt{1 + \sigma(x)}$

$\simeq v\left(1 + \frac{1}{2}\,[\,\sigma(x) + i\,\theta(x)\,]\right)$ (lowest order in σ, θ)

$\phi(x) = v + \eta(x)$

→ $\eta(x) \simeq \frac{1}{2}\, v\,[\,\sigma(x) + i\,\theta(x)\,]$

$\eta^+(x) \simeq \frac{1}{2}\, v\,[\,\sigma(x) - i\,\theta(x)\,]$

Conversely,

$\sigma(x) = \frac{1}{v}[\,\eta^+(x) + \eta(x)\,]$

$\theta(x) = i\,\frac{1}{v}[\,\eta^+(x) - \eta(x)\,]$

$[\,\Theta(x),\, \rho(x')\,]_{t=t'} = -i\,\delta(\mathbf{r} - \mathbf{r}')$

→ $-i\,\delta(\mathbf{r} - \mathbf{r}') = \left[\frac{\theta(x)}{2},\; v^2[1 + \sigma(x')\,]\right]_{t=t'}$

$= \frac{v^2}{2}[\theta(x),\; \sigma(x')\,]_{t=t'}$

or $[\sigma(x),\ \theta(x')]_{t=t'} = \frac{2i}{v^2}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$

$$\eta(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\eta_{\boldsymbol{k}}\,e^{-ikx} = \int \frac{d^3k}{(2\pi)^{3/2}}\,\eta_{\boldsymbol{k}}(t)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

where $k\,x = \omega_{\boldsymbol{k}}\,t - \boldsymbol{k}\cdot\boldsymbol{r}$ $\qquad \omega_{\boldsymbol{k}} = \frac{\epsilon_{\boldsymbol{k}}}{\hbar} = \frac{\hbar\,\boldsymbol{k}^2}{2m}$

and $\eta_{\boldsymbol{k}}(t) = \eta_{\boldsymbol{k}}\,e^{-i\omega_{\boldsymbol{k}}t}$ (interaction picture)

→ $\eta^+(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\eta^+_{\boldsymbol{k}}\,e^{ikx} = \int \frac{d^3k}{(2\pi)^{3/2}}\,\eta^+_{\boldsymbol{k}}(t)\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}$

$$\eta^+_{\boldsymbol{k}}(t) = \eta^+_{\boldsymbol{k}}\,e^{i\omega_{\boldsymbol{k}}t}$$

$$\sigma(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\sigma_{\boldsymbol{k}}(t)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$$\sigma^+(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\sigma^+_{\boldsymbol{k}}(t)\,e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} = \int \frac{d^3k}{(2\pi)^{3/2}}\,\sigma^+_{-\boldsymbol{k}}(t)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

$\sigma^+(x) = \sigma(x)$ → $\sigma_{\boldsymbol{k}}(t) = \sigma^+_{-\boldsymbol{k}}(t)$

$$[\sigma(x),\ \sigma^+(x')]_{t=t'} = [\sigma(x),\ \sigma(x')]_{t=t'} = [\sigma^+(x),\ \sigma^+(x')]_{t=t'} = 0$$

→ $[\sigma_{\boldsymbol{k}}(t),\ \sigma_{\boldsymbol{k}'}(t)] = [\sigma_{\boldsymbol{k}}(t),\ \sigma^+_{\boldsymbol{k}'}(t)] = [\sigma^+_{\boldsymbol{k}}(t),\ \sigma^+_{\boldsymbol{k}'}(t)] = 0$

Similarly,

$$\theta(x) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\theta_{\boldsymbol{k}}(t)\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$$

with $\theta_{\boldsymbol{k}}(t) = \theta^+_{-\boldsymbol{k}}(t)$

and $[\theta_{\boldsymbol{k}}(t),\ \theta_{\boldsymbol{k}'}(t)] = [\theta_{\boldsymbol{k}}(t),\ \theta^+_{\boldsymbol{k}'}(t)] = [\theta^+_{\boldsymbol{k}}(t),\ \theta^+_{\boldsymbol{k}'}(t)] = 0.$

$$[\sigma(x),\ \theta(x')]_{t=t'} = \frac{2i}{v^2}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

→ $\int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}}\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}+i\boldsymbol{k}'\cdot\boldsymbol{r}'}\,[\sigma_{\boldsymbol{k}}(t),\ \theta_{\boldsymbol{k}'}(t)] = \frac{2i}{v^2}\int \frac{d^3k}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$

∴ $[\sigma_{\boldsymbol{k}}(t),\ \theta_{\boldsymbol{k}'}(t)] = \frac{2i}{v^2}\,\delta(\boldsymbol{k}+\boldsymbol{k}')$

$$\eta(x) \simeq \frac{1}{2}\,v\,[\,\sigma(x) + i\,\theta(x)\,]$$

→ $\eta_{\boldsymbol{k}}(t) \simeq \frac{1}{2}\,v\,[\,\sigma_{\boldsymbol{k}}(t) + i\,\theta_{\boldsymbol{k}}(t)\,]$

$$\eta^+_{\boldsymbol{k}}(t) \simeq \frac{1}{2}\,v\,[\,\sigma^+_{\boldsymbol{k}}(t) - i\,\theta^+_{\boldsymbol{k}}(t)\,] = \frac{1}{2}\,v\,[\,\sigma_{-\boldsymbol{k}}(t) - i\,\theta_{-\boldsymbol{k}}(t)\,]$$

Thus,

$$[\,\eta_{\boldsymbol{k}}(t),\ \eta^+_{\boldsymbol{k}}(t)\,] = [\,\eta_{\boldsymbol{k}},\ \eta^+_{\boldsymbol{k}}\,] = \frac{v^2}{4}[\,\sigma_{\boldsymbol{k}}(t) + i\,\theta_{\boldsymbol{k}}(t),\ \sigma^+_{\boldsymbol{k}}(t) - i\,\theta^+_{\boldsymbol{k}}(t)\,]$$

$$= \frac{v^2}{4}\{\,[\,\sigma_{\boldsymbol{k}}(t),\ -i\,\theta^+_{\boldsymbol{k}}(t)\,] + [\,i\,\theta_{\boldsymbol{k}}(t),\ \sigma^+_{\boldsymbol{k}}(t)\,]\,\}$$

$$= \frac{v^2}{4}\{\,[\,\sigma_{\boldsymbol{k}}(t),\ \theta_{-\boldsymbol{k}}(t)\,] - [\,\theta_{\boldsymbol{k}}(t),\ \sigma_{-\boldsymbol{k}}(t)\,]\,\}$$

$$= \frac{v^2}{4}(-i)\frac{2i}{v^2}\{\,\delta(0) + \delta(0)\}$$
$$= \delta(0)$$

as expected.

Note:

Since $\xi_{\boldsymbol{k}}^{+}(t)\,\xi_{\boldsymbol{k}}(t) = \xi_{-\boldsymbol{k}}(t)\,\xi_{\boldsymbol{k}}(t) = \xi_{\boldsymbol{k}}^{+}\,\xi_{\boldsymbol{k}} = \xi_{\boldsymbol{k}}\,\xi_{\boldsymbol{k}}^{+} = \;\ldots$ for $\xi = \sigma,\ \theta$,

we can drop the time dependence in the following calculations for $\mathcal{H}_{\text{free}}(\boldsymbol{k})$.

$$\eta_{\boldsymbol{k}}^{+}\,\eta_{\boldsymbol{k}} = \frac{v^2}{4}\left(\sigma_{\boldsymbol{k}}^{+} - i\,\theta_{\boldsymbol{k}}^{+}\right)(\sigma_{\boldsymbol{k}} + i\,\theta_{\boldsymbol{k}})$$
$$= \frac{v^2}{4}\left[\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}} + i\left(\sigma_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}}\right)\right]$$
$$= \frac{v^2}{4}\left[\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}} - \frac{2}{v^2}\,\delta(0)\right]$$
$$= \frac{v^2}{4}\left(\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}}\right) - \frac{1}{2}\,\delta(0)$$

$$\eta_{\boldsymbol{k}}\,\eta_{-\boldsymbol{k}} = \frac{v^2}{4}(\sigma_{\boldsymbol{k}} + i\,\theta_{\boldsymbol{k}})(\sigma_{-\boldsymbol{k}} + i\,\theta_{-\boldsymbol{k}})$$
$$= \frac{v^2}{4}\left[\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} - \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}} + i\,(\sigma_{\boldsymbol{k}}\,\theta_{-\boldsymbol{k}} + \theta_{\boldsymbol{k}}\,\sigma_{-\boldsymbol{k}})\right]$$

$$\eta_{\boldsymbol{k}}^{+}\,\eta_{-\boldsymbol{k}}^{+} = \frac{v^2}{4}(\sigma_{\boldsymbol{k}} - i\,\theta_{\boldsymbol{k}})(\sigma_{-\boldsymbol{k}} - i\,\theta_{-\boldsymbol{k}})$$
$$= \frac{v^2}{4}\left[\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} - \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}} - i\,(\sigma_{\boldsymbol{k}}\,\theta_{-\boldsymbol{k}} + \theta_{\boldsymbol{k}}\,\sigma_{-\boldsymbol{k}})\right]$$

$$\therefore \qquad \mathcal{H}_{\text{free}}(\boldsymbol{k}) = (\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\,\eta_{\boldsymbol{k}}^{+}\,\eta_{\boldsymbol{k}} + U_{\boldsymbol{k}}\left(\eta_{\boldsymbol{k}}\,\eta_{-\boldsymbol{k}} + \eta_{\boldsymbol{k}}^{+}\,\eta_{-\boldsymbol{k}}^{+}\right) + \left(\frac{1}{2}\,\varepsilon_{\boldsymbol{k}} + U_{\boldsymbol{k}}\right)\delta(0)$$
$$= \frac{v^2}{4}\left[(\varepsilon_{\boldsymbol{k}} + 2\,U_{\boldsymbol{k}})\left(\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}}\right) + 2\,U_k\left(\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} - \theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}}\right)\right]$$
$$= \frac{v^2}{4}\left[(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})\,\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + \varepsilon_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}}\right]$$
$$= A_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}^{+}\,\sigma_{\boldsymbol{k}} + B_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^{+}\,\theta_{\boldsymbol{k}}$$

where

$$B_{\boldsymbol{k}} = \frac{v^2}{4}\,\varepsilon_{\boldsymbol{k}}$$
$$A_{\boldsymbol{k}} = \frac{v^2}{4}(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})$$

Using $\qquad \varepsilon_{\boldsymbol{k}} = \dfrac{\hbar^2\,\boldsymbol{k}^2}{2\,m} \qquad\qquad E_{\boldsymbol{k}} = \sqrt{\varepsilon_{\boldsymbol{k}}\,(\varepsilon_{\boldsymbol{k}} + 4\,U_{\boldsymbol{k}})}$

we have

$$B_{\boldsymbol{k}} = v^2\,\frac{\hbar^2\,\boldsymbol{k}^2}{8\,m}$$
$$A_{\boldsymbol{k}} = \frac{v^2}{4}\,\frac{E_{\boldsymbol{k}}^2}{\varepsilon_{\boldsymbol{k}}} = \frac{v^2}{2}\,\frac{m\,E_{\boldsymbol{k}}^2}{\hbar^2\,\boldsymbol{k}^2}$$

Setting $\dfrac{1}{M_{\boldsymbol{k}}} = v^2 \dfrac{\hbar^2 \boldsymbol{k}^2}{4\,m}$, we have

$$B_{\boldsymbol{k}} = \frac{1}{2\,M_{\boldsymbol{k}}}$$

$$A_{\boldsymbol{k}} = \frac{v^4}{8}\,M_{\boldsymbol{k}}\,E_{\boldsymbol{k}}^2$$

Despite the appearance,

$$\begin{aligned}\mathcal{H}_{\text{free}}(\boldsymbol{k}) &= A_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}^+\,\sigma_{\boldsymbol{k}} + B_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^+\,\theta_{\boldsymbol{k}} \\ &= \frac{1}{2\,M_{\boldsymbol{k}}}\,\theta_{\boldsymbol{k}}^+\,\theta_{\boldsymbol{k}} + \frac{v^4}{8}\,M_{\boldsymbol{k}}\,E_{\boldsymbol{k}}^2\,\sigma_{\boldsymbol{k}}^+\,\sigma_{\boldsymbol{k}}\end{aligned}$$

is not diagonal since $[\,\sigma_{\boldsymbol{k}}\,,\ \theta_{\boldsymbol{k}}\,] \neq 0$.

Setting

$$\chi_{\boldsymbol{k}} = \sqrt{A_{\boldsymbol{k}}}\ \sigma_{\boldsymbol{k}} + i\,\sqrt{B_{\boldsymbol{k}}}\ \theta_{\boldsymbol{k}} \qquad\qquad \chi_{\boldsymbol{k}}^+ = \sqrt{A_{\boldsymbol{k}}}\ \sigma_{\boldsymbol{k}}^+ - i\,\sqrt{B_{\boldsymbol{k}}}\ \theta_{\boldsymbol{k}}^+$$

$$\rightarrow \qquad \chi_{\boldsymbol{k}}^+\,\chi_{\boldsymbol{k}} = A_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}^+\,\sigma_{\boldsymbol{k}} + B_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^+\,\theta_{\boldsymbol{k}} + i\,\sqrt{A_{\boldsymbol{k}}\,B_{\boldsymbol{k}}}\ \left(\sigma_{\boldsymbol{k}}^+\,\theta_{\boldsymbol{k}} - \theta_{\boldsymbol{k}}^+\,\sigma_{\boldsymbol{k}}\right)$$

$$\chi_{\boldsymbol{k}}\,\chi_{\boldsymbol{k}}^+ = A_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}^+ + B_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^+ + i\,\sqrt{A_{\boldsymbol{k}}\,B_{\boldsymbol{k}}}\ \left(-\sigma_{\boldsymbol{k}}\,\theta_{\boldsymbol{k}}^+ + \theta_{\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}^+\right)$$

$$[\sigma_{\boldsymbol{k}}\,(t),\ \theta_{\boldsymbol{k}'}(t)] = \frac{2\,i}{v^2}\,\delta(\boldsymbol{k} + \boldsymbol{k}')$$

$$\begin{aligned}\rightarrow \qquad [\,\chi_{\boldsymbol{k}}\,,\ \chi_{\boldsymbol{k}}^+\,] &= 2\,i\,\sqrt{A_{\boldsymbol{k}}\,B_{\boldsymbol{k}}}\ \left([\theta_{\boldsymbol{k}},\ \sigma_{\boldsymbol{k}}^+] + [\theta_{\boldsymbol{k}}^+,\ \sigma_{\boldsymbol{k}}]\right) \\ &= \frac{4}{v^2}\,\sqrt{A_{\boldsymbol{k}}\,B_{\boldsymbol{k}}}\ \delta(0) = \frac{4}{v^2}\,\sqrt{\frac{v^4\,E_{\boldsymbol{k}}^2}{16}}\ \delta(0) = E_{\boldsymbol{k}}\,\delta(0)\end{aligned}$$

$$\chi_{\boldsymbol{k}}^+\,\chi_{\boldsymbol{k}} + \chi_{\boldsymbol{k}}\,\chi_{\boldsymbol{k}}^+ = 2\,\mathcal{H}_{\text{free}}(\boldsymbol{k}) + f_{\boldsymbol{k}}$$

where $\qquad f_{\boldsymbol{k}} = i\,\sqrt{A_{\boldsymbol{k}}\,B_{\boldsymbol{k}}}\ \left(-\sigma_{\boldsymbol{k}}\,\theta_{-\boldsymbol{k}} + \theta_{\boldsymbol{k}}\,\sigma_{-\boldsymbol{k}} + \sigma_{-\boldsymbol{k}}\,\theta_{\boldsymbol{k}} - \theta_{-\boldsymbol{k}}\,\sigma_{\boldsymbol{k}}\right)$

Since $f_{-\boldsymbol{k}} = -f_{\boldsymbol{k}}$, it doesn't contribute to

$$H_{\text{free}} = \int d^3\,k\,\mathcal{H}_{\text{free}}(\boldsymbol{k})$$

Dropping $f_{\boldsymbol{k}}$, we have

$$\begin{aligned}\mathcal{H}_{\text{free}}(\boldsymbol{k}) &= \frac{1}{2}\left(\chi_{\boldsymbol{k}}^+\,\chi_{\boldsymbol{k}} + \chi_{\boldsymbol{k}}\,\chi_{\boldsymbol{k}}^+\right) \\ &= \chi_{\boldsymbol{k}}^+\,\chi_{\boldsymbol{k}} + \frac{1}{2}\,E_{\boldsymbol{k}}\,\delta(0)\end{aligned}$$

Setting $\zeta_{\boldsymbol{k}} = \dfrac{1}{\sqrt{E_{\boldsymbol{k}}}}\,\chi_{\boldsymbol{k}}$

we have $\qquad [\,\zeta_{\boldsymbol{k}}\,,\ \zeta_{\boldsymbol{k}'}^+\,] = \delta(\boldsymbol{k} - \boldsymbol{k}')$

and $\qquad \mathcal{H}_{\text{free}}(\boldsymbol{k}) = E_{\boldsymbol{k}}\left[\,\zeta_{\boldsymbol{k}}^+\,\zeta_{\boldsymbol{k}} + \dfrac{1}{2}\,\delta(0)\,\right] \qquad$ (diagonal)

Finally,

$$B_{\boldsymbol{k}} = \frac{1}{2\,M_{\boldsymbol{k}}} \qquad\qquad A_{\boldsymbol{k}} = \frac{v^4}{8}\,M_{\boldsymbol{k}}\,E_{\boldsymbol{k}}^2$$

$$\rightarrow \qquad \zeta_{\mathbf{k}} = \frac{1}{\sqrt{E_{\mathbf{k}}}}\chi_{\mathbf{k}} = \sqrt{\frac{A_{\mathbf{k}}}{E_{\mathbf{k}}}}\,\sigma_{\mathbf{k}} + i\sqrt{\frac{B_{\mathbf{k}}}{E_{\mathbf{k}}}}\,\theta_{\mathbf{k}}$$

$$= \frac{v^2}{2}\sqrt{\frac{M_k E_{\mathbf{k}}}{2}}\,\sigma_{\mathbf{k}} + \frac{i}{\sqrt{2 M_{\mathbf{k}} E_{\mathbf{k}}}}\,\theta_{\mathbf{k}}$$

$$= \frac{v}{2}\left(\sqrt{G_{\mathbf{k}}}\,\sigma_{\mathbf{k}} + \frac{i}{\sqrt{G_{\mathbf{k}}}}\,\theta_{\mathbf{k}}\right) \quad \text{where} \qquad G_{\mathbf{k}} = \frac{1}{2} v^2 M_k E_{\mathbf{k}}$$

Squeezed State

See Summary section of 1.6._SqueezedCoherentState.pdf .

Comparing

$$\mathcal{H}_{\text{free}}(\mathbf{k}) = \frac{1}{2 M_{\mathbf{k}}}\theta_{\mathbf{k}}^{+}\theta_{\mathbf{k}} + \frac{v^4}{8} M_{\mathbf{k}} E_{\mathbf{k}}^2 \sigma_{\mathbf{k}}^{+}\sigma_{\mathbf{k}}$$

$$= \frac{1}{2} E_{\mathbf{k}}\left(\zeta_{\mathbf{k}}^{+}\zeta_{\mathbf{k}} + \zeta_{\mathbf{k}}\zeta_{\mathbf{k}}^{+}\right)$$

$$[\sigma_{\mathbf{k}}(t), \theta_{-\mathbf{k}}(t)] = \frac{2 i}{v^2}\delta(0) = \frac{2 i}{v^2}\frac{V}{(2\pi)^3}$$

with the standard quadratic Hamiltonian

$$H = \frac{1}{2 M} p^2 + \frac{1}{2} M \omega^2 q^2 = \frac{1}{2}\hbar\omega\left(a^{+} a + a a^{+}\right)$$

$$[q, p] = i\hbar \qquad \text{and} \qquad a = \frac{1}{\sqrt{2 M \hbar\omega}}(M\omega q + i p)$$

we have

$$\sigma_{\mathbf{k}} = \sqrt{\frac{V}{(2\pi)^3}\frac{2}{v^2\hbar}}\,q_{\mathbf{k}} \qquad \theta_{\mathbf{k}} = \sqrt{\frac{V}{(2\pi)^3}}\,p_{-\mathbf{k}}$$

so that

$$[q_{\mathbf{k}}, p_{\mathbf{k}}] = i\hbar$$

$$\mathcal{H}_{\text{free}}(\mathbf{k}) = \frac{V}{(2\pi)^3}\left(\frac{1}{2 M_{\mathbf{k}}} p_{\mathbf{k}} p_{-\mathbf{k}} + \frac{1}{2} M_{\mathbf{k}}\omega_{\mathbf{k}}^2 q_{\mathbf{k}} q_{-\mathbf{k}}\right)$$

where $\qquad E_{\mathbf{k}} = \hbar\omega_{\mathbf{k}}$

Also,

$$\zeta_{\mathbf{k}} = \frac{v^2}{2}\sqrt{\frac{M_k E_{\mathbf{k}}}{2}}\,\sigma_{\mathbf{k}} + \frac{i}{\sqrt{2 M_{\mathbf{k}} E_{\mathbf{k}}}}\,\theta_{\mathbf{k}}$$

$$= \sqrt{\frac{V}{(2\pi)^3}}\frac{1}{\sqrt{2 M_{\mathbf{k}}\hbar\omega_{\mathbf{k}}}}\left(M_{\mathbf{k}}\omega_{\mathbf{k}} q_{\mathbf{k}} + i p_{-\mathbf{k}}\right)$$

$$= \sqrt{\frac{V}{(2\pi)^3}}\,a_{\mathbf{k}}$$

→ $\mathcal{H}_{\text{free}}(\boldsymbol{k}) = \frac{V}{(2\pi)^3} \frac{1}{2} \left(a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}} + a_{\boldsymbol{k}} a_{\boldsymbol{k}}^{+}\right)$

According to 1.6._SqueezedCoherentState.pdf , one can make a Bogoliubov transformation to some squeezed state $|\, v \rangle\rangle$ such that

$$\langle\langle (\Delta q_{\boldsymbol{k}})^2 \rangle\rangle = \frac{(2\pi)^3}{V} \frac{\hbar}{2 M_{\boldsymbol{k}} \omega_{\boldsymbol{k}}} e^{-2\tau_k}$$

$$\langle\langle (\Delta p_{\boldsymbol{k}})^2 \rangle\rangle = \frac{(2\pi)^3}{V} \frac{M_{\boldsymbol{k}} \hbar \omega_{\boldsymbol{k}}}{2} e^{2\tau_k}$$

$$\langle\langle (\Delta q_{\boldsymbol{k}})^2 \rangle\rangle \langle\langle (\Delta p_{\boldsymbol{k}})^2 \rangle\rangle = \left(\frac{(2\pi)^3}{V}\right)^2 \frac{\hbar^2}{4}$$

With $\sigma_{\boldsymbol{k}} = \sqrt{\frac{V}{(2\pi)^3} \frac{2}{v^2 \hbar}}\, q_{\boldsymbol{k}}$ $\qquad \theta_{\boldsymbol{k}} = \sqrt{\frac{V}{(2\pi)^3}}\, p_{-\boldsymbol{k}}$

we have

$$\langle\langle (\Delta \sigma_{\boldsymbol{k}})^2 \rangle\rangle = \frac{2}{v^4 M_{\boldsymbol{k}} \hbar \omega_{\boldsymbol{k}}} e^{-2\tau_k}$$

$$\langle\langle (\Delta p_{\boldsymbol{k}})^2 \rangle\rangle = \frac{M_{\boldsymbol{k}} \hbar \omega_{\boldsymbol{k}}}{2} e^{2\tau_k}$$

$$\langle\langle (\Delta q_{\boldsymbol{k}})^2 \rangle\rangle \langle\langle (\Delta p_{\boldsymbol{k}})^2 \rangle\rangle = \frac{1}{v^4}$$