4.6. Superfluidity

Schrodinger (classical) field ,

$$\mathcal{H} = \frac{\hbar^2}{2m} \nabla\phi^* \cdot \nabla\phi + \frac{1}{2} g \left(\phi^* \phi - v^2\right)^2$$

Minimum of potential is at

$$\phi^*(\boldsymbol{r})\, \phi(\boldsymbol{r}) = v^2$$

or

$$\phi(\boldsymbol{r}) = v\, e^{i\alpha} \qquad v,\ \alpha = \text{real constants}$$

Consider the field

$$\phi(\boldsymbol{r}) = v\, e^{i\chi(\boldsymbol{r})} \qquad \chi \text{ real}$$

$$\rightarrow \qquad \phi^*(\boldsymbol{r})\, \phi(\boldsymbol{r}) = v^2$$

$$\nabla\phi(\boldsymbol{r}) = v\,(i\nabla\chi)\, e^{i\chi(\boldsymbol{r})} = i\,\phi\,\nabla\chi$$

$$\therefore \qquad \nabla\phi^* \cdot \nabla\phi = (-i\,\phi^*\,\nabla\chi)\cdot(i\,\phi\,\nabla\chi) = v^2\,\nabla\chi\cdot\nabla\chi$$

$$\mathcal{H} = \frac{\hbar^2 v^2}{2m} \nabla\chi\cdot\nabla\chi$$

$$\boldsymbol{j}(\boldsymbol{r}) = \frac{\hbar}{2mi}\left[\phi^*\,\nabla\phi - (\nabla\phi^*)\,\phi\right]$$

$$= \frac{\hbar}{2mi}\left[\phi^*\, i\,\phi\,\nabla\chi - (-i\,\phi^*\,\nabla\chi)\,\phi\right]$$

$$= \frac{\hbar v^2}{m} \nabla\chi$$

When ϕ is quantized, the ground state is given by

$$\langle\, \phi(\boldsymbol{r})\, \rangle = v \qquad (\ \alpha = 0 \text{ for convenience }\ )$$

$$\langle\mathcal{H}\rangle = \frac{\hbar^2 v^2}{2m} \nabla\chi\cdot\nabla\chi$$

$$\langle \boldsymbol{j}(\boldsymbol{r})\rangle = \frac{\hbar v^2}{m} \nabla\chi \qquad (\text{ phase current })$$

When a particle is in the Bose condensate, symmetryis spontaneously broken and a massless Goldstone boson appears (see §4.7). If the Goldstone bosons do not interact with any particle, they behave like a superfluid.

Consider a model hamiltonian of particles & phonons (Goldstone bosons) :

$$H = \int d^3K\, \frac{\hbar^2 \boldsymbol{K}^2}{2M}\, a_{\boldsymbol{K}}^+ a_{\boldsymbol{K}} + \int d^3k\, E_{\boldsymbol{k}}\, \zeta_{\boldsymbol{k}}^+ \zeta_{\boldsymbol{k}} + H_{\text{int}}$$

where $\qquad E_{\boldsymbol{k}} = C\,\hbar\, |\,\boldsymbol{k}\,|$

and $\qquad H_{\text{int}} = g \int d^3r\, \phi^+(\boldsymbol{r})\, \phi(\boldsymbol{r})\, \zeta(\boldsymbol{r})$

Plane wave expansion:

$$\phi(\boldsymbol{r}) = \int \frac{d^3K}{(2\pi)^{3/2}}\, a_{\boldsymbol{K}}\, e^{i\boldsymbol{K}\cdot\boldsymbol{r}}$$

$$\zeta(\boldsymbol{r}) = c \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}} \left(\zeta_{\boldsymbol{k}}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \zeta_{\boldsymbol{k}}^+\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\right)$$

Note: phonon is massless & hence treated relativistically.

$$H_{\text{int}} = g \int d^3 r \int \frac{d^3 K}{(2\pi)^{3/2}} \int \frac{d^3 K'}{(2\pi)^{3/2}} \int \frac{d^3 k}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}}$$

$$\times\, a_{\boldsymbol{K}}^{+} a_{\boldsymbol{K'}} \left(\zeta_{\boldsymbol{k}} e^{i(-\boldsymbol{K}+\boldsymbol{K'}+\boldsymbol{k})\cdot \boldsymbol{r}} + \zeta_{\boldsymbol{k}}^{+} e^{i(-\boldsymbol{K}+\boldsymbol{K'}-\boldsymbol{k})\cdot \boldsymbol{r}} \right.$$

$$= \frac{g}{(2\pi)^{3/2}\sqrt{2\hbar\omega_{\boldsymbol{k}}}} \int d^3 K \int d^3 k$$

$$\times \left(a_{\boldsymbol{K}}^{+} a_{\boldsymbol{K}-\boldsymbol{k}} \zeta_{\boldsymbol{k}} + a_{\boldsymbol{K}}^{+} a_{\boldsymbol{K}+\boldsymbol{k}} \zeta_{\boldsymbol{k}}^{+} \right)$$

Term $a_{\boldsymbol{K}}^{+} a_{\boldsymbol{K}-\boldsymbol{k}} \zeta_{\boldsymbol{k}}$ describes absorption of phonon with

$$\hbar(\boldsymbol{K}-\boldsymbol{k}) + \hbar\boldsymbol{k} = \hbar\boldsymbol{K} \qquad (\boldsymbol{p} \text{ conservation})$$

$$\frac{\hbar^2(\boldsymbol{K}-\boldsymbol{k})^2}{2M} + C\hbar\,|\boldsymbol{k}| = \frac{\hbar^2 \boldsymbol{K}^2}{2M} \qquad (E \text{ conservation})$$

$$\rightarrow \qquad \frac{C}{\hbar}|\boldsymbol{k}| = \frac{\boldsymbol{K}^2-(\boldsymbol{K}-\boldsymbol{k})^2}{2M} = \frac{2\boldsymbol{K}\cdot\boldsymbol{k}-\boldsymbol{k}^2}{2M}$$

$$\leq \frac{2\boldsymbol{K}\cdot\boldsymbol{k}}{2M} = \frac{V_o}{\hbar}|\boldsymbol{k}|\cos\theta$$

$$\leq \frac{V_o}{\hbar}|\boldsymbol{k}|$$

where $V_o = \frac{\hbar\,|\boldsymbol{K}|}{M}$ is the velocity of the outgoing particle.

$\therefore \qquad V_o \geq C \qquad$ for phonon absorption

Term $a_{\boldsymbol{K}}^{+} a_{\boldsymbol{K}+\boldsymbol{k}} \zeta_{\boldsymbol{k}}^{+}$ describes emission of phonon with

$$\hbar(\boldsymbol{K}+\boldsymbol{k}) = \hbar\boldsymbol{k} + \hbar\boldsymbol{K} \qquad (\boldsymbol{p} \text{ conservation})$$

$$\frac{\hbar^2(\boldsymbol{K}+\boldsymbol{k})^2}{2M} = C\hbar\,|\boldsymbol{k}| + \frac{\hbar^2 \boldsymbol{K}^2}{2M} \qquad (E \text{ conservation})$$

$$\rightarrow \qquad \frac{C}{\hbar}|\boldsymbol{k}| = \frac{(\boldsymbol{K}+\boldsymbol{k})^2-\boldsymbol{K}^2}{2M} = \frac{\boldsymbol{K}'^2-(\boldsymbol{K}'-\boldsymbol{k})^2}{2M}$$

where $\boldsymbol{K}' = \boldsymbol{K}+\boldsymbol{k}$ is the momentum of the incoming particle.

$$\therefore \qquad \frac{C}{\hbar}|\boldsymbol{k}| = \frac{2\boldsymbol{K}'\cdot\boldsymbol{k}-\boldsymbol{k}^2}{2M}$$

$$\leq \frac{\boldsymbol{K}'\cdot\boldsymbol{k}}{M} = \frac{V_i}{\hbar}|\boldsymbol{k}|\cos\theta$$

$$\leq \frac{V_i}{\hbar}|\boldsymbol{k}|$$

where $V_i = \frac{\hbar\,|\boldsymbol{K}'|}{M}$ is the velocity of the incoming particle.

$\therefore \qquad V_i \geq C \qquad$ for phonon emission

For low enough temperatures, phonon absorptions can be neglected.
If the particles are bosons, then $V << \mathrm{C}$ in the Bose condensate so that phonon emissions are forbidden. Hence, superfluidity.