4.7. Goldstone Theorem

Bose condensation & spontaneous symmetry breaking

→ $\langle 0 \mid \phi(x) \mid 0 \rangle = v \neq 0$ (ground state)

with $\langle \underline{0} \mid \phi(x) \mid \underline{0} \rangle = 0$ (false vacuum)

Let H be invariant under a continuous global transformation.
Spontaneous symmetry breaking → gapless Goldstone mode (boson).

See the original paper,

J.Goldstone, A.Salam, S.Weinberg, PR 127, 965 (62)

for a more general discussion.

Phase Transformation

Let Q be the generator of a phase transformation:

$$e^{-i\alpha Q}\, \phi(x)\, e^{i\alpha Q} = e^{i\alpha}\, \phi(x) \qquad \alpha = \text{real}$$

For $|\alpha| << 1$,

$$(1 - i\alpha Q)\, \phi(x)\, (1 + i\alpha Q) = (1 + i\alpha)\, \phi(x)$$

→ $[\phi(x),\ Q] = \phi(x)$

If H is invariant under the phase transformation, then

$$Q = \frac{1}{c}\int d^3x\, j^0(x) \qquad (\, j^\mu = \text{Noether current}\,)$$

is conserved, i.e., $\dfrac{dQ}{dt} = 0.$

Ground state is not invariant under the phase transformation since

$$|\alpha\rangle \equiv e^{i\alpha Q}\left|0\right\rangle \neq \left|0\right\rangle \quad \text{is another ground state}$$

$$\langle \alpha \mid \phi(x) \mid \alpha \rangle = \left\langle 0 \right| e^{-i\alpha Q}\, \phi(x)\, e^{i\alpha Q} \left| 0 \right\rangle$$

$$= e^{i\alpha} \langle \alpha \mid \phi(x) \mid \alpha \rangle$$

$$= e^{i\alpha}\, v$$

$$e^{i\alpha Q}\left|0\right\rangle \neq \left|0\right\rangle$$

$|\alpha| << 1$ → $(1 + i\alpha Q)\,|0\rangle \neq |0\rangle$

∴ $Q\,|0\rangle \neq 0$

$\phi(x)\,|\underline{0}\rangle = 0$ → $Q\,|\underline{0}\rangle = 0$

Proof

$$\langle 0 \mid [\phi(x),\ Q] \mid 0 \rangle = \int d^3y \left\langle 0 \right| [\phi(x),\ j^0(y)] \left| 0 \right\rangle$$

$$= \langle 0 \mid \phi(x) \mid 0 \rangle = v \neq 0$$

→ $I_+ - I_- = v$

where

$$I_+ = \int d^3y \left\langle 0 \right| \phi(x)\, j^0(y) \left| 0 \right\rangle$$

$$I_- = \int d^3y \left\langle 0 \right| j^0(y)\, \phi(x) \left| 0 \right\rangle$$

Completeness:

$$\int d\xi \int dE \int d^3k \;\; | E, \boldsymbol{k}, \xi \rangle\langle E, \boldsymbol{k}, \xi | = I$$

Translational invariance

→ $\phi(x) = e^{-\frac{i}{\hbar} p\cdot x}\, \phi(0)\, e^{\frac{i}{\hbar} p\cdot x}$

$j^0(y) = e^{-\frac{i}{\hbar} p\cdot y}\, j^0(0)\, e^{\frac{i}{\hbar} p\cdot y}$

∴ $$I_+ = \int d\xi \int dE \int d^3k \int d^3y \langle 0 \,|\, \phi(x) \;| E, \boldsymbol{k}, \xi \rangle\langle E, \boldsymbol{k}, \xi \,|\, j^0(y) \,\big|\, 0 \big\rangle$$

$$= \int d\xi \int dE \int d^3k \int d^3y \, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}) - \frac{i}{\hbar} E\Delta t} \langle 0 \,|\, \phi(0) \;|\; E, \boldsymbol{k}, \xi \rangle \big\langle E, \boldsymbol{k}, \xi \,|\, j^0(0) \,\big|\, 0 \big\rangle$$

$$= \int dE \int d^3k \int d^3y \, e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}) - \frac{i}{\hbar} E\Delta t} \sigma_+ (\boldsymbol{k}, E)$$

where

$\Delta t = t_x - t_y$

$\sigma_+(\boldsymbol{k}, E) = \int d\xi \langle 0 \,|\, \phi(0) \;|\, E, \boldsymbol{k}, \xi \rangle \big\langle E, \boldsymbol{k}, \xi \,|\, j^0(0) \,\big|\, 0 \big\rangle$ (spectral function)

$$I_+ = (2\pi)^3 \int dE \int d^3k \, \delta(\boldsymbol{k})\, e^{-\frac{i}{\hbar} E\Delta t} \sigma_+(\boldsymbol{k}, E)$$

$$= (2\pi)^3 \int dE \;\; e^{-\frac{i}{\hbar} E\Delta t} \sigma_+(\boldsymbol{0}, E)$$

Similarly,

$$I_- = (2\pi)^3 \int dE \;\; e^{+\frac{i}{\hbar} E\Delta t} \sigma_-(\boldsymbol{0}, E)$$

with $\sigma_-(\boldsymbol{k}, E) = \int d\xi \big\langle 0 \,\big|\, j^0(0) \;\big|\; E, \boldsymbol{k}, \xi \big\rangle\langle E, \boldsymbol{k}, \xi \,|\, \phi(0) \,|\, 0 \rangle$

$I_+ - I_- = v$

→ $$(2\pi)^3 \int dE \;\; e^{-\frac{i}{\hbar} E\Delta t} [\, \sigma_+(\boldsymbol{0}, E) - \sigma_-(\boldsymbol{0}, E) \,] = v$$

$$(2\pi)^4 [\, \sigma_+(\boldsymbol{0}, E) - \sigma_-(\boldsymbol{0}, E) \,] = v \int \frac{d\,\Delta t}{\hbar} \;\; e^{\frac{i}{\hbar} E\Delta t}$$

$$\sigma_+(\boldsymbol{0}, E) - \sigma_-(\boldsymbol{0}, E) = \frac{v}{(2\pi)^3} \, \delta(E)$$

→ At least one of σ_+ & σ_- contains a $\delta(E)$ factor.

∴ ∃ (gapless) mode with $E = 0$, which occurs at $\boldsymbol{k} = 0$.