5.1. Maxwell Equations

Warning: Gaussian units are used here.
Also, our metric tensor is the negative of that used by Ezawa.

Ref: J.D.Jackson,"Classical Electrodynamics".

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x^\mu = (c\,t,\ \boldsymbol{r}) \qquad\qquad x_\mu = (c\,t,\ -\boldsymbol{r})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\partial_t,\ \nabla\right) \qquad\qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\partial_t,\ -\nabla\right)$$

Field Tensors

Maxwell eqs:

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{B} \qquad\qquad \nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{H} = \frac{1}{c}\partial_t\boldsymbol{D} + \frac{4\pi}{c}\boldsymbol{J} \qquad\qquad \nabla\cdot\boldsymbol{D} = 4\pi\rho_e$$

Linear media:

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} \qquad\qquad \boldsymbol{B} = \mu_m\boldsymbol{H}$$

The homogeneous eqs are satisfied if

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \qquad\qquad \boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{A} - \nabla\varphi$$

or

$$A^\mu = (\varphi,\ \boldsymbol{A}) \qquad\qquad A_\mu = (\varphi,\ -\boldsymbol{A})$$

Note:
Bold faced quantities are 3-vectors with no distinction between co- & contra- variant components, e.g.,

$$\boldsymbol{A}_i = \boldsymbol{A}^i = A^i = -A_i \qquad\qquad i = 1, 2, 3$$

The totally antisymmetric symbol $\varepsilon_{ijk} = \varepsilon^{ijk} = \varepsilon_i{}^{jk} = \ \ldots$ is not a tensor.
For consistency, we'll always use the subscript forms, i.e., $\boldsymbol{A}_i$ & ε_{ijk}.

Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\rightarrow \qquad F_{ij} = \partial_i A_j - \partial_j A_i = \varepsilon_{ijk}\,\varepsilon_{klm}\,\partial_l A_m$$

$$= -\varepsilon_{ijk}\,(\nabla\times\boldsymbol{A})_k = -\varepsilon_{ijk}\,B_k$$

$$F_{0i} = \partial_0 A_i - \partial_i A_0 = -\frac{1}{c}\partial_t\boldsymbol{A}_i - \nabla\varphi$$

$$= \boldsymbol{E}_i = -F_{i0}$$

$$\therefore \qquad F_{\mu\nu} = \begin{pmatrix} 0 & \boldsymbol{E}_1 & \boldsymbol{E}_2 & \boldsymbol{E}_3 \\ -\boldsymbol{E}_1 & 0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ -\boldsymbol{E}_2 & \boldsymbol{B}_3 & 0 & -\boldsymbol{B}_1 \\ -\boldsymbol{E}_3 & -\boldsymbol{B}_2 & \boldsymbol{B}_1 & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \eta^{\mu\sigma}\,\eta^{\nu\tau}\,F_{\sigma\tau}$$
$$= \begin{pmatrix} 0 & -\boldsymbol{E}_1 & -\boldsymbol{E}_2 & -\boldsymbol{E}_3 \\ \boldsymbol{E}_1 & 0 & -\boldsymbol{B}_3 & \boldsymbol{B}_2 \\ \boldsymbol{E}_2 & \boldsymbol{B}_3 & 0 & -\boldsymbol{B}_1 \\ \boldsymbol{E}_3 & -\boldsymbol{B}_2 & \boldsymbol{B}_1 & 0 \end{pmatrix}$$
$$F^{ij} = \varepsilon_{ijk}\,\varepsilon_{klm}\,\partial^l A^m = -\varepsilon_{ijk}(\nabla\times\boldsymbol{A})_k = -\varepsilon_{ijk}\,\boldsymbol{B}_k$$
$$F^{0i} = \frac{1}{c}\,\partial_t \boldsymbol{A}_i + \nabla\varphi = -\boldsymbol{E}_i = -F^{i0}$$
$$\varepsilon_{ijl}\,F^{ij} = -\varepsilon_{ijl}\,\varepsilon_{ijk}\,\boldsymbol{B}_k = -(\delta_{jj}\,\delta_{lk} - \delta_{jk}\,\delta_{lj})\,\boldsymbol{B}_k$$
$$= -(3\,\boldsymbol{B}_l - \boldsymbol{B}_l) = -2\,\boldsymbol{B}_l$$

$\therefore$
$$\boldsymbol{B}_i = -\frac{1}{2}\,\varepsilon_{ijk}\,F^{jk}$$
$$\boldsymbol{E}_i = F^{i0}$$

Let
$$\mathcal{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\sigma\tau}\,F_{\sigma\tau} = \begin{pmatrix} 0 & -\boldsymbol{B}_1 & -\boldsymbol{B}_2 & -\boldsymbol{B}_3 \\ \boldsymbol{B}_1 & 0 & \boldsymbol{E}_3 & -\boldsymbol{E}_2 \\ \boldsymbol{B}_2 & -\boldsymbol{E}_3 & 0 & \boldsymbol{E}_1 \\ \boldsymbol{B}_3 & \boldsymbol{E}_2 & -\boldsymbol{E}_1 & 0 \end{pmatrix} \qquad \text{(dual of } F^{\mu\nu}\text{)}$$

$\rightarrow$ $\partial_\mu \mathcal{F}^{\mu\nu} = 0$

or $\partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0$ (Homogeneous Maxwell eqs.)

Let
$$G^{\mu\nu} = F^{\mu\nu}(\boldsymbol{E}\to\boldsymbol{D},\ \boldsymbol{B}\to\boldsymbol{H}) = \begin{pmatrix} 0 & -\boldsymbol{D}_1 & -\boldsymbol{D}_2 & -\boldsymbol{D}_3 \\ \boldsymbol{D}_1 & 0 & -\boldsymbol{H}_3 & \boldsymbol{H}_2 \\ \boldsymbol{D}_2 & \boldsymbol{H}_3 & 0 & -\boldsymbol{H}_1 \\ \boldsymbol{D}_3 & -\boldsymbol{H}_2 & \boldsymbol{H}_1 & 0 \end{pmatrix}$$
$$J^\mu = (c\,\rho_e\,,\ \boldsymbol{J})$$

$\rightarrow$ $\partial_\mu G^{\mu\nu} = \dfrac{4\pi}{c}\,J^\nu$ (Inhomogeneous Maxwell eqs.)

Vacuum with Free Charges

In vacuum, all charges are considered free, $G_{\mu\nu} = F_{\mu\nu}$ & $K_{\mu\nu} = 0$.
Dynamical variables for the EM degrees of freedom are A^μ.
Dynamical variables for the particles are J_f^μ.

Let
$$\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{c}\,J_f^\mu\,A_\mu$$
$$\frac{\partial\,F^{\alpha\beta}}{\partial\,\partial^\mu A^\nu} = \frac{\partial\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right)}{\partial\,\partial^\mu A^\nu} = \delta_\mu^\alpha\,\delta_\nu^\beta - \delta_\mu^\beta\,\delta_\nu^\alpha$$
$$\frac{\partial\,F^{\alpha\beta}\,F_{\alpha\beta}}{\partial\,\partial^\mu A^\nu} = \eta_{\alpha\gamma}\,\eta_{\beta\sigma}\,\frac{\partial\,F^{\alpha\beta}\,F^{\gamma\sigma}}{\partial\,\partial^\mu A^\nu}$$
$$= \eta_{\alpha\gamma}\,\eta_{\beta\sigma}\left[\left(\delta_\mu^\alpha\,\delta_\nu^\beta - \delta_\mu^\beta\,\delta_\nu^\alpha\right)F^{\gamma\sigma} + F^{\alpha\beta}\left(\delta_\mu^\gamma\,\delta_\nu^\sigma - \delta_\mu^\sigma\,\delta_\nu^\gamma\right)\right]$$
$$= \left(\eta_{\mu\gamma}\,\eta_{\nu\sigma} - \eta_{\nu\gamma}\,\eta_{\mu\sigma}\right)F^{\gamma\sigma} + F^{\alpha\beta}\left(\eta_{\alpha\mu}\,\eta_{\beta\nu} - \eta_{\alpha\nu}\,\eta_{\beta\mu}\right)$$
$$= F_{\mu\nu} - F_{\nu\mu} + F_{\mu\nu} - F_{\nu\mu}$$
$$= 4\,F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial\, \partial^\mu A^\nu} = -\frac{1}{4\pi} F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -\frac{1}{c} J_{f\,\nu}$$

E-L eq.:

$$-\frac{1}{c} J_{f\,\nu} + \frac{1}{4\pi} \partial^\mu F_{\mu\nu} = 0$$

or $$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J_f^\nu$$

$$\partial_\nu F^{\nu 0} = \frac{4\pi}{c} J_f^0 = \partial_i F^{i\,0}$$
$$= \nabla \cdot \boldsymbol{E} = 4\pi \rho_f$$

$$\partial_\nu F^{\nu\, i} = \frac{4\pi}{c} J_f^i = \frac{1}{c} \partial_t F^{0\,i} + \partial_j F^{j\,i}$$
$$= -\frac{1}{c} \partial_t \boldsymbol{E}_i - \varepsilon_{j\,i\,k}\, \partial_j \boldsymbol{B}_k$$
$$= -\frac{1}{c} \partial_t \boldsymbol{E}_i + (\nabla \times \boldsymbol{B})_i \;= \frac{4\pi}{c} \boldsymbol{J}_{f\,i}$$

→ $$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{J}_f + \frac{1}{c} \partial_t \boldsymbol{E}$$

$$F_{\mu\nu} F^{\mu\nu} = F_{0\,i} F^{0\,i} + F_{i\,0} F^{i\,0} + F_{i\,j} F^{i\,j}$$
$$F_{i\,j} F^{i\,j} = \varepsilon_{i\,j\,k} \boldsymbol{B}_k\, \varepsilon_{i\,j\,l} \boldsymbol{B}_l \;= \left(\delta_{j\,j}\, \delta_{k\,l} - \delta_{j\,l}\, \delta_{k\,j}\right) \boldsymbol{B}_k \boldsymbol{B}_l$$
$$= 3\boldsymbol{B}^2 - \boldsymbol{B}^2 \;= 2\boldsymbol{B}^2$$
$$F_{0\,i} F^{0\,i} = F_{i\,0} F^{i\,0} = -\boldsymbol{E}^2$$

∴ $$F_{\mu\nu} F^{\mu\nu} = 2\left(\boldsymbol{B}^2 - \boldsymbol{E}^2\right)$$

$$\frac{1}{c} J_f^\mu A_\mu = \rho_f\, \varphi + \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A}$$

∴ $$\mathcal{L} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) - \rho_f\, \varphi - \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A}$$

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_t A^\mu} = \frac{\partial \mathcal{L}}{c\, \partial\, \partial^0 A^\mu} = -\frac{1}{4\pi c} F_{0\,\mu}$$
$$= \left(0, -\frac{1}{4\pi c} \boldsymbol{E}\right)$$

$$\pi_\mu\, \partial_t A^\mu = -\frac{1}{4\pi c} \boldsymbol{E} \cdot \partial_t \boldsymbol{A}$$
$$= \frac{1}{4\pi} \boldsymbol{E} \cdot (\boldsymbol{E} + \nabla \varphi)$$
$$= \frac{1}{4\pi} \boldsymbol{E}^2 + \frac{1}{4\pi} [\nabla \cdot (\varphi \boldsymbol{E}) - \varphi \nabla \cdot \boldsymbol{E}]$$
$$= \frac{1}{4\pi} \boldsymbol{E}^2 - \rho_f\, \varphi + \frac{1}{4\pi} \nabla \cdot (\varphi \boldsymbol{E})$$

$$\mathcal{H} = \pi_\mu\, \partial_t A^\mu - \mathcal{L}$$
$$= \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) + \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A} + \frac{1}{4\pi} \nabla \cdot (\varphi \boldsymbol{E})$$

$$H = \int d^3 r\, \mathcal{H} = \int d^3 r \left[\frac{1}{8\pi} \left(\boldsymbol{E}^2 + \boldsymbol{B}^2 \right) + \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A} \right]$$

Linear Medium

The Lagrangian that can give both the correct Euler eq. & H for a general medium is suprisingly difficult to write down.

For a linear medium in which

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} \quad \& \qquad \boldsymbol{B} = \mu_m \boldsymbol{H}$$

we have

$$S = \int d t \int d^3 x\, \mathcal{L} = \frac{1}{c} \int d^4 x\, \mathcal{L}$$

$$\mathcal{L} = \mathcal{L}_{\text{EM}} - \frac{1}{c} J_f^{\mu} A_{\mu}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} \left[\varepsilon \left(F_{0i} F^{0i} + F_{i0} F^{i0} \right) + \frac{1}{\mu_m} F_{ij} F^{ij} \right]$$

$$= -\frac{1}{16\pi} \left(2 \varepsilon F_{0i} F^{0i} + \frac{1}{\mu_m} F_{ij} F^{ij} \right)$$

$$\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \partial_0 A_i} = -\frac{1}{16\pi} \varepsilon \left(2 F^{0i} - 2 F^{i0} \right) = -\frac{1}{4\pi} \varepsilon F^{0i} = -\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \partial_i A_0}$$

$$\frac{\partial \mathcal{L}_{\text{EM}}}{\partial \partial_i A_j} = -\frac{1}{4\pi} \frac{1}{\mu_m} F^{ij}$$

$$\frac{\partial \mathcal{L}}{\partial A_{\mu}} = -\frac{1}{c} J_f^{\mu}$$

E-L eq.:

$$\frac{\partial \mathcal{L}}{\partial A_{\nu}} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial \partial_{\mu} A_{\nu}} = 0$$

$\nu = 0$:

$$\frac{\partial \mathcal{L}}{\partial A_0} - \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i A_0} = 0 \qquad \text{since} \qquad \frac{\partial \mathcal{L}}{\partial \partial_0 A_0} = 0$$

$$-\frac{1}{c} J_f^0 - \frac{1}{4\pi} \varepsilon \partial_i F^{0i} = 0$$

$$-\rho_f + \frac{1}{4\pi} \varepsilon \nabla \cdot \boldsymbol{E} = 0$$

$$\nabla \cdot \boldsymbol{D} = 4\pi \rho_f$$

$\nu = i$:

$$\frac{\partial \mathcal{L}}{\partial A_i} - \partial_0 \frac{\partial \mathcal{L}}{\partial \partial_0 A_i} - \partial_j \frac{\partial \mathcal{L}}{\partial \partial_j A_i} = 0$$

$$-\frac{1}{c} J_f^i + \frac{1}{4\pi} \left(\varepsilon \partial_0 F^{0i} + \frac{1}{\mu_m} \partial_j F^{ji} \right) = 0$$

$$-\frac{1}{c} \boldsymbol{J}_f + \frac{1}{4\pi} \left(-\varepsilon \frac{\partial \boldsymbol{E}}{c\, \partial t} + \frac{1}{\mu_m} \nabla \times \boldsymbol{B} \right) = 0$$

$$\nabla\times\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}_f + \frac{\partial \boldsymbol{D}}{c\,\partial t}$$

$$F_{0i}F^{0i} = F_{i0}F^{i0} = -\boldsymbol{E}^2 \qquad F_{ij}F^{ij} = 2\,\boldsymbol{B}^2$$

$$\frac{1}{c}J_f^{\mu}A_{\mu} = \rho_f\,\varphi + \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A}$$

$$\rightarrow \qquad \mathcal{L} = \frac{1}{8\pi}\left(\varepsilon\,\boldsymbol{E}^2 - \frac{1}{\mu_m}\boldsymbol{B}^2\right) - \rho_f\,\varphi - \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A}$$

$$= \frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D} - \boldsymbol{B}\cdot\boldsymbol{H}) - \rho_f\,\varphi - \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A}$$

$$\pi_{\mu} = \frac{\partial\mathcal{L}}{\partial\,\partial_t A^{\mu}} = \frac{\partial\mathcal{L}}{c\,\partial\,\partial^0 A^{\mu}} = -\frac{1}{4\pi}\varepsilon\,F_{0\mu}$$

$$= \left(0, -\frac{1}{4\pi c}\varepsilon\,\boldsymbol{E}\right) = \left(0, -\frac{1}{4\pi c}\boldsymbol{D}\right)$$

$$\pi_{\mu}\,\partial_t A^{\mu} = -\frac{1}{4\pi c}\boldsymbol{D}\cdot\partial_t\boldsymbol{A}$$

$$= \frac{1}{4\pi}\boldsymbol{D}\cdot(\boldsymbol{E} + \nabla\varphi)$$

$$= \frac{1}{4\pi}\boldsymbol{D}\cdot\boldsymbol{E} + \frac{1}{4\pi}[\nabla\cdot(\varphi\,\boldsymbol{D}) - \varphi\,\nabla\cdot\boldsymbol{D}]$$

$$= \frac{1}{4\pi}\boldsymbol{D}\cdot\boldsymbol{E} - \rho_f\,\varphi + \frac{1}{4\pi}\nabla\cdot(\varphi\,\boldsymbol{D})$$

$$\mathcal{H} = \pi_{\mu}\,\partial_t A^{\mu} - \mathcal{L}$$

$$= \frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) + \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A} + \frac{1}{4\pi}\nabla\cdot(\varphi\,\boldsymbol{D})$$

$$= \frac{1}{8\pi}\left(\varepsilon\,\boldsymbol{E}^2 + \frac{1}{\mu_m}\boldsymbol{B}^2\right) + \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A} + \frac{1}{4\pi}\nabla\cdot(\varphi\,\varepsilon\,\boldsymbol{E})$$

$$H = \int d^3r\,\mathcal{H} = \int d^3r\left[\frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}) + \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A}\right]$$

$$= \int d^3r\left[\frac{1}{8\pi}\left(\varepsilon\,\boldsymbol{E}^2 + \frac{1}{\mu_m}\boldsymbol{B}^2\right) + \frac{1}{c}\boldsymbol{J}_f\cdot\boldsymbol{A}\right]$$

$$\varepsilon\,\nabla\cdot\boldsymbol{E} = 4\pi\,\rho_f$$

$$\boldsymbol{E} = -\nabla\phi \qquad \rightarrow \qquad \varepsilon\,\nabla^2\phi = -4\pi\,\rho_f$$

$$\nabla^2\frac{1}{|\,\boldsymbol{r} - \boldsymbol{r}'\,|} = -4\pi\,\delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\rightarrow \qquad \phi(t, \boldsymbol{r}) = \int d^3r'\,\frac{\rho_f(t, \boldsymbol{r}')}{\varepsilon\,|\,r - r'\,|}$$

$$\boldsymbol{E}(t, \boldsymbol{r}) = -\nabla\int d^3r'\,\frac{\rho_f(t, \boldsymbol{r}')}{\varepsilon\,|\,\boldsymbol{r} - \boldsymbol{r}'\,|}$$

$$\int d^3r\,\boldsymbol{E}^2 = \int d^3r\;\nabla\phi\cdot\nabla\phi$$

$$= \int d^3r\,[\nabla\cdot(\phi\,\nabla\phi) - \phi\,\nabla^2\phi]$$

$$= -\int d^3 r\, \phi \nabla^2 \phi$$

$$= \frac{4\pi}{\varepsilon} \int d^3 r\, \phi\, \rho_f$$

$$= \frac{4\pi}{\varepsilon^2} \int d^3 r \int d^3 r' \frac{\rho_f(t, \boldsymbol{r})\, \rho_f(t, \boldsymbol{r}')}{|\ \boldsymbol{r} - \boldsymbol{r}'\ |}$$

$$\therefore \qquad H = \int d^3 r \left[\frac{1}{2} \int d^3 r' \frac{\rho_f(t, \boldsymbol{r})\, \rho_f(t, \boldsymbol{r}')}{\varepsilon\, |\ \boldsymbol{r} - \boldsymbol{r}'\ |} + \frac{1}{8\pi\mu_m} \boldsymbol{B}^2 + \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A} \right]$$

$$\mathcal{H} = \frac{1}{2} \int d^3 r' \frac{\rho_f(t, \boldsymbol{r})\, \rho_f(t, \boldsymbol{r}')}{\varepsilon\, |\ \boldsymbol{r} - \boldsymbol{r}'\ |} + \frac{1}{8\pi\mu_m} \boldsymbol{B}^2 + \frac{1}{c} \boldsymbol{J}_f \cdot \boldsymbol{A}$$

where the term $\frac{1}{4\pi} \nabla \cdot (\varphi\, \varepsilon\, \boldsymbol{E})$ is dropped.

Gauge Transformation

See J.D.Jackson, "Classical Electrodynamics", 2nd ed., §§ 6.4-5.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Under a gauge transformation

$$A_\mu \;\to\; A'_\mu = A_\mu + \partial_\mu f$$

i.e.,

$$-\boldsymbol{A} \;\to\; -\boldsymbol{A}' = -\boldsymbol{A} + \nabla f$$

$$\varphi \;\to\; \varphi' = \varphi + \frac{1}{c} \frac{\partial f}{\partial t}$$

we have

$$\partial_\mu A_\nu \;\to\; \partial_\mu (A_\nu + \partial_\nu f) = \partial_\mu A_\nu + \partial_\mu \partial_\nu f$$

$$\partial_\nu A_\mu \;\to\; \partial_\nu (A_\mu + \partial_\mu f) = \partial_\nu A_\mu + \partial_\nu \partial_\mu f$$

$$\therefore \qquad F_{\mu\nu} \;\to\; F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}$$

i.e., the EM fields are invariant under a gauge transformation.

Thus, the description of the EM fields in terms of A_μ is not unique.

To make it unique (apart from additive constants), we need to specify f.

Since only $\partial_\mu f$ appears in the transformation, this tantamounts to specifying a 2nd order partial differential equation that f satisfies.

Such a specification is called fixing the gauge.

Alternatively, we can say that the components of A are not independent. Fixing the gauge is equivalent to choosing how the dependency is removed.

Lorentz Gauge

$$\partial_\mu A^\mu \;\to\; \partial_\mu A^\mu + \partial_\mu \partial^\mu f$$

$\therefore \qquad \partial_\mu A^\mu = \frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A}$ is invariant if we restrict to those f for which

$$\partial_\mu \partial^\mu f = 0 = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) f$$

Choosing f to satisfy $\partial_\mu \partial^\mu f = 0$, or equivalently,

choosing A^μ to satisfy the Lorentz condition

$$\partial_\mu A^\mu = 0$$

gives us the Lorentz gauge.

Note that the Lorentz condition itself is covariant, i.e., invariant under the Lorentz transformation of special relativity.

Coulomb Gauge

$$-\boldsymbol{A} \to -\boldsymbol{A}' = -\boldsymbol{A} + \nabla f$$

$$\nabla\cdot\boldsymbol{A} \to \nabla\cdot\boldsymbol{A}' = \nabla\cdot\boldsymbol{A} - \nabla^2 f$$

∴ $\nabla\cdot\boldsymbol{A}$ is invariant if we restrict to those f for which $\nabla^2 f = 0$.

Choosing f to satisfy $\nabla^2 f = 0$, or equivalently,
choosing A^μ to satisfy

$$\nabla\cdot\boldsymbol{A} = 0$$

gives us the Coulomb (also called transverse or radiation) gauge.

Since

$$\boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{A} - \nabla\varphi \qquad \& \qquad \nabla\cdot\boldsymbol{E} = 4\pi\rho_{\text{tot}}$$

we see that

$$\nabla^2\varphi = -4\pi\rho_{\text{tot}}$$

$$\to \qquad \varphi(t,\boldsymbol{r}) = \int d^3r' \frac{\rho_{\text{tot}}(t,\boldsymbol{r}')}{|\ \boldsymbol{r}-\boldsymbol{r}'\ |} \qquad \text{(instantaneous Coulomb potential)}$$

$$= \int d^3r' \frac{\rho_f(t,\boldsymbol{r}')}{\varepsilon\,|\ \boldsymbol{r}-\boldsymbol{r}'\ |} \qquad \text{for linear media}$$

Eqs for A^μ in Vacuum

$$\mathcal{L} = \mathcal{L}_{\text{EM}} - \frac{1}{c} J_f^\mu A_\mu$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$$

$$= -\frac{1}{16\pi}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$= -\frac{1}{8\pi}\left(\partial_\mu A_\nu\cdot\partial^\mu A^\nu - \partial_\nu A_\mu\cdot\partial^\mu A^\nu\right)$$

$$= -\frac{1}{8\pi}\left(\partial_\mu A_\nu\cdot\partial^\mu A^\nu + A_\mu\,\partial^\mu\partial_\nu A^\nu\right) + \frac{1}{8\pi}\partial_\nu\left(A_\mu\,\partial^\mu A^\nu\right)$$

$$= -\frac{1}{8\pi}\left(\partial_\mu A_\nu\cdot\partial^\mu A^\nu + A_\mu\,\partial^\mu\partial_\nu A^\nu\right)$$

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J_f^\nu \qquad J_f^\mu = (c\,\rho_f\,,\ \boldsymbol{J}_f) \qquad \text{(free current)}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\to \qquad \partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{4\pi}{c} J_f^\nu$$

$$= \partial_\mu\partial^\mu A^\nu - \partial^\nu\left(\partial_\mu A^\mu\right)$$

$$= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\nu - \partial^\nu\left(\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right)$$

For $v = 0$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi - \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = 4\pi\rho_f$$

$$-\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) = 4\pi\rho_f$$

For $v = j$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{A} + \nabla\left(\frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \frac{4\pi}{c}\boldsymbol{J}_f$$

Eqs for A^μ in Linear Medium

$$\mathcal{L} = \mathcal{L}_{EM} - \frac{1}{c} J_f^\mu A_\mu$$

$$\mathcal{L}_{EM} = -\frac{1}{16\pi}\left(2\,\varepsilon F_{0i}F^{0i} + \frac{1}{\mu_m}F_{ij}F^{ij}\right)$$

$$F_{0i}F^{0i} = (\partial_0 A_i - \partial_i A_0)\left(\partial^0 A^i - \partial^i A^0\right)$$
$$= \partial_0 A_i\cdot\partial^0 A^i + \partial_i A_0\cdot\partial^i A^0 - \partial_0 A_i\,\partial^i A^0 - \partial_i A_0\,\partial^0 A^i$$
$$= \partial_0 A_i\cdot\partial^0 A^i + \partial_i A_0\cdot\partial^i A^0 - 2\,\partial_0 A_i\,\partial^i A^0$$

$$2F_{0i}F^{0i} = F_{0i}F^{0i} + F_{i0}F^{i0}$$
$$= 2\left(\partial_0 A_i\cdot\partial^0 A^i + \partial_i A_0\cdot\partial^i A^0\right) - 4\,\partial_0 A_i\,\partial^i A^0$$

$$F_{ij}F^{ij} = \left(\partial_i A_j - \partial_j A_i\right)\left(\partial^i A^j - \partial^j A^i\right)$$
$$= 2\left(\partial_i A_j\cdot\partial^i A^j - \partial_i A_j\cdot\partial^j A^i\right)$$
$$= 2\left(\partial_i A_j\cdot\partial^i A^j + A_j\,\partial^j\partial_i A^i\right) - 2\,\partial_i\left(A_j\,\partial^j A^i\right)$$

$$\mathcal{L}_{EM} = -\frac{1}{8\pi}\left(\varepsilon\left(\partial_0 A_i\cdot\partial^0 A^i + \partial_i A_0\cdot\partial^i A^0\right) + \frac{1}{\mu_m}\left(\partial_i A_j\cdot\partial^i A^j + A_j\,\partial^j\partial_i A^i\right)\right.$$
$$\left. - 2\,\varepsilon\,\partial_0 A_i\,\partial^i A^0 - \frac{1}{\mu_m}\partial_i\left(A_j\,\partial^j A^i\right)\right)$$
$$= -\frac{1}{8\pi}\left(\varepsilon\left(\partial_0 A_i\cdot\partial^0 A^i + \partial_i A_0\cdot\partial^i A^0\right) + \frac{1}{\mu_m}\left(\partial_i A_j\cdot\partial^i A^j + A_j\,\partial^j\partial_i A^i\right)\right)$$
$$= -\frac{1}{8\pi\mu_m}\left(\varepsilon\,\mu_m\,\partial_0 A_i\cdot\partial^0 A^i + \partial_j A_i\cdot\partial^j A^i\right)$$
$$-\frac{1}{8\pi}\left(\varepsilon\,\partial_i A_0\cdot\partial^i A^0 + \frac{1}{\mu_m}A_j\,\partial^j\partial_i A^i\right)$$

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c}J_f^\nu$$

or $\quad \partial_0 G^{0\nu} + \partial_i G^{i\nu} = \dfrac{4\pi}{c}J_f^\nu \qquad\qquad J_f^\mu = (c\,\rho_f\,,\ \boldsymbol{J}_f)$ (free current)

$$G^{0i} = -\boldsymbol{D}_i = -\varepsilon\,\boldsymbol{E}_i = -G^{i0} = \varepsilon\,F^{0i}$$

$$G^{ij} = -\varepsilon_{ijk}\boldsymbol{H}_k = -\frac{1}{\mu_m}\varepsilon_{ijk}\boldsymbol{B}_k = \frac{1}{\mu_m}F^{ij}$$

$v = 0$: $\qquad \partial_i G^{i0} = 4\pi\rho_f = \nabla\cdot\boldsymbol{D}$

→ $\qquad \varepsilon\,\partial_i F^{i0} = 4\pi\rho_f$

$$\partial_i\left(\partial^i A^0 - \partial^0 A^i\right) = \frac{4\,\pi\,\rho_f}{\varepsilon}$$

$$-\nabla^2 \varphi - \frac{1}{c}\,\frac{\partial}{\partial t}\,(\nabla \cdot \boldsymbol{A}) = \frac{4\,\pi\,\rho_f}{\varepsilon}$$

$v = j$:

$$\partial_0 G^{0\,j} + \partial_i G^{i\,j} = \frac{4\,\pi}{c}\,J_f^j$$

→

$$\varepsilon\,\partial_0 F^{0\,j} + \frac{1}{\mu_m}\,\partial_i F^{i\,j} = \frac{4\,\pi}{c}\,J_f^j$$

$$\varepsilon\,\partial_0\left(\partial^0 A^j - \partial^j A^0\right) + \frac{1}{\mu_m}\,\partial_i\left(\partial^i A^j - \partial^j A^i\right) = \frac{4\,\pi}{c}\,J_f^j$$

$$\varepsilon\left(\frac{1}{c^2}\,\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{1}{c}\,\frac{\partial\,\nabla\varphi}{\partial t}\right) + \frac{1}{\mu_m}\left[-\nabla^2 \boldsymbol{A} + \nabla\,(\nabla \cdot \boldsymbol{A})\right] = \frac{4\,\pi}{c}\,\boldsymbol{J}_f$$

$$\frac{\varepsilon\,\mu_m}{c^2}\,\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla\left(\frac{\varepsilon\,\mu_m}{c}\,\frac{\partial\,\varphi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = \frac{4\,\pi}{c}\,\mu_m\,\boldsymbol{J}_f$$