5.2.a. Canonical Quantization : Vacuum, Coulomb Gauge

Basics

In vacuum with $J^{\mu} = (\rho c, \ \boldsymbol{J}) = 0$:

$$\partial_{\mu} \partial^{\mu} A^{\nu} - \partial^{\nu} (\partial_{\mu} A^{\mu}) = 0$$

or

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A^{\nu} - \partial^{\nu} \left(\frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A} \right) = 0$$

i.e.,

$$-\nabla^2 \varphi - \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \boldsymbol{A}) = 0$$

$$\frac{1}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} + \nabla \left(\frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \boldsymbol{A} \right) = 0$$

$$\begin{aligned} \mathcal{L}_{\text{EM}} &= -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{16\pi} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}) \\ &= -\frac{1}{8\pi} (\partial_{\mu} A_{\nu} \cdot \partial^{\mu} A^{\nu} - \partial_{\nu} A_{\mu} \cdot \partial^{\mu} A^{\nu}) \\ &= -\frac{1}{8\pi} (\partial_{\mu} A_{\nu} \cdot \partial^{\mu} A^{\nu} + A_{\mu} \partial^{\mu} \partial_{\nu} A^{\nu}) + \frac{1}{8\pi} \partial_{\nu} (A_{\mu} \partial^{\mu} A^{\nu}) \\ &= -\frac{1}{8\pi} (\partial_{\mu} A_{\nu} \cdot \partial^{\mu} A^{\nu} + A_{\mu} \partial^{\mu} \partial_{\nu} A^{\nu}) \end{aligned}$$

$$\pi_{\mu} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial \dot{A}^{\mu}} = \frac{\partial \mathcal{L}_{\text{EM}}}{c\, \partial\, \partial^0 A^{\mu}}$$

$$\mathcal{H}_{\text{EM}} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial \dot{A}^{\mu}} \dot{A}^{\mu} - \mathcal{L}_{\text{EM}} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial\, \partial^0 A^{\mu}} \partial^0 A^{\mu} - \mathcal{L}_{\text{EM}}$$

Energy momentum tensor (see J.D.Jackson, “Classical Electrodynamics”, 2nd ed., § 12.10) :

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial\, \partial_{\mu} A^{\lambda}} \partial^{\nu} A^{\lambda} - \eta^{\mu\nu} \mathcal{L}_{\text{EM}}$$

$$\therefore \qquad \Theta^{00} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial\, \partial_0 A^{\lambda}} \partial^0 A^{\lambda} - \mathcal{L}_{\text{EM}} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial \dot{A}^{\lambda}} \dot{A}^{\lambda} - \mathcal{L}_{\text{EM}}$$

$$= \pi_{\lambda} \dot{A}^{\lambda} - \mathcal{L}_{\text{EM}} = \mathcal{H}_{\text{EM}}$$

$$\Theta^{0j} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial\, \partial_0 A^{\lambda}} \partial^j A^{\lambda}$$

Coulomb Gauge in Vacuum

See M.Kaku, “Quantum Field Theory”, §4.3.

$$A^0 = \varphi = 0 \qquad\qquad \nabla \cdot \boldsymbol{A} = \partial_i A^i = 0$$

$\rightarrow \qquad \nabla^2 \varphi = 0$ (automatically satisfied)

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = 0$$

$$\mathcal{L}_{EM} = -\frac{1}{8\pi}\partial_\mu A_i \cdot \partial^\mu A^i = \frac{1}{8\pi}\sum_{i=1}^{3}(\partial_\mu \boldsymbol{A}_i)\partial^\mu \boldsymbol{A}_i$$

Comparing with the real Klein-Gordon field (see 3.3._RealKlein-GordonField.pdf) where

$$\mathcal{L} = \frac{1}{2} f \left(\partial_\mu \phi \cdot \partial^\mu \phi - \frac{m^2 c^2}{\hbar^2}\phi^2 \right)$$

we see that each $A^i = \boldsymbol{A}_i$ can be treated as a massless K-G field with

$$f = \frac{1}{4\pi}$$

Comparing with

$$\phi(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k}} e^{-i k\cdot x} + a^+_{\boldsymbol{k}} e^{i k\cdot x} \right)$$

a plane wave expansion of $\boldsymbol{A}$ thus take the form

$$\boldsymbol{A}(x) = \hbar c\sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\boldsymbol{k}}}} \left(\boldsymbol{a}_{\boldsymbol{k}} e^{-i k\cdot x} + \boldsymbol{a}^+_{\boldsymbol{k}} e^{i k\cdot x} \right)$$

where $\boldsymbol{a}_{\boldsymbol{k}}$ is now a vector.

$$\nabla\cdot\boldsymbol{A}(x) = \hbar c\sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\boldsymbol{k}}}}\, i\,\boldsymbol{k}\cdot \left(\boldsymbol{a}_{\boldsymbol{k}} e^{-i k\cdot x} - \boldsymbol{a}^+_{\boldsymbol{k}} e^{i k\cdot x} \right)$$

$$\nabla\cdot\boldsymbol{A} = 0 \qquad \rightarrow \qquad \boldsymbol{k}\cdot\boldsymbol{a}_{\boldsymbol{k}} = 0 \qquad \forall\ \boldsymbol{k}$$

Let $\boldsymbol{\epsilon}^\lambda(\boldsymbol{k})$, $\lambda = 1, 2$, be two orthonormal vectors in the plane perpendicular to $\boldsymbol{k}$.
Condition $\boldsymbol{k}\cdot\boldsymbol{a}_{\boldsymbol{k}} = 0$ is satisfied if

$$\boldsymbol{a}_{\boldsymbol{k}} = \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^\lambda(\boldsymbol{k})\, a^\lambda_{\boldsymbol{k}}$$

$\boldsymbol{\epsilon}^\lambda(\boldsymbol{k})$ are called the polarization vectors.

$$\therefore \qquad \boldsymbol{A}(x) = \hbar c\sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\hbar\omega_{\boldsymbol{k}}}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^\lambda(\boldsymbol{k}) \left(a^\lambda_{\boldsymbol{k}} e^{-i k\cdot x} + a^{\lambda +}_{\boldsymbol{k}} e^{i k\cdot x} \right)$$

To quantize the EM field, one simply set

$$\left[a^\lambda_{\boldsymbol{k}},\, a^{+\lambda'}_{\boldsymbol{k}'} \right] = \delta^{\lambda\lambda'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')$$

while

$$\left[a^\lambda_{\boldsymbol{k}},\, a^{\lambda'}_{\boldsymbol{k}'} \right] = \left[a^{+\lambda}_{\boldsymbol{k}},\, a^{+\lambda'}_{\boldsymbol{k}'} \right] = 0$$

Alternatively, one may want to start with

$$\pi_j = \frac{\partial \mathcal{L}_{EM}}{\partial \dot{A}^j} = \frac{\partial \mathcal{L}_{EM}}{c\,\partial\,\partial^0 A^j} = -\frac{1}{4\pi}F_{0j} = -\frac{1}{4\pi}\boldsymbol{E}_j$$

$$= -\frac{1}{4\pi c}\partial_0 A_j = \frac{1}{4\pi c^2}\dot{\boldsymbol{A}}_j$$

& quantize the fields according to

$$\left[\boldsymbol{A}_i(t, \boldsymbol{r}),\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right] = i\hbar\,\delta_{ij}\,\delta(\boldsymbol{r}-\boldsymbol{r}')$$

However,

$$\partial_i \left[\boldsymbol{A}_i(t, \boldsymbol{r}),\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right] = \nabla\cdot\left[\boldsymbol{A}(t, \boldsymbol{r}),\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right]$$

$$= \left[\nabla \cdot \boldsymbol{A}(t, \boldsymbol{r}) \,,\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right] = 0$$

but $\quad i \hbar \partial_i \delta_{ij} \delta(\boldsymbol{r} - \boldsymbol{r}') = i \hbar \partial_j \delta(\boldsymbol{r} - \boldsymbol{r}') \neq 0$

so that the commutator relation is incompatible with the gauge condition.

The correct quantization is (see Kaku)

$$\left[\boldsymbol{A}_i(t, \boldsymbol{r}) \,,\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right] = i \hbar \overline{\delta}_{ij}(\boldsymbol{r} - \boldsymbol{r}')$$

or $\quad \left[\boldsymbol{A}_i(t, \boldsymbol{r}) \,,\ \dot{\boldsymbol{A}}_j(t, \boldsymbol{r}') \right] = 4 \pi c^2 i \hbar \overline{\delta}_{ij}(\boldsymbol{r} - \boldsymbol{r}')$

where

$$\begin{aligned} \overline{\delta}_{ij}(\boldsymbol{r} - \boldsymbol{r}') &= \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \left(\delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \right) \\ &= \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \\ &= \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(\boldsymbol{r} - \boldsymbol{r}') \\ &= \delta_{ij} \delta(\boldsymbol{r} - \boldsymbol{r}') - \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \end{aligned}$$

For $i \neq j$,

$$\begin{aligned} \overline{\delta}_{ij}(\boldsymbol{r} - \boldsymbol{r}') &= - \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \\ &= -\frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \left[e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} + e^{-i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \right] \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \\ &= 0 \end{aligned}$$

since the integrand is odd.

Also

$$\begin{aligned} \partial_i \overline{\delta}_{ij}(\boldsymbol{r} - \boldsymbol{r}') &= i \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \boldsymbol{k}_i \left(\delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \right) \\ &= i \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \left(\boldsymbol{k}_j - \frac{\boldsymbol{k}^2 \boldsymbol{k}_j}{\boldsymbol{k}^2} \right) \\ &= 0 \end{aligned}$$

as advertized.

The relation

$$\boldsymbol{k}_i \left(\delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \right) = 0$$

shows that $\delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2}$ is a projector onto the plane perpendicular to $\boldsymbol{k}$.

The completeness condition that $\left\{ \boldsymbol{\epsilon}^\lambda(\boldsymbol{k}) \,,\ \lambda = 1, 2 \right\}$ spans the plane perpendicular to $\boldsymbol{k}$ is therefore

$$\sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^\lambda(\boldsymbol{k}) \, \boldsymbol{\epsilon}^\lambda(\boldsymbol{k})^T = I - \frac{\boldsymbol{k} \boldsymbol{k}^T}{\boldsymbol{k}^T \boldsymbol{k}} \qquad \text{(matrix form)}$$

or $\quad \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}_i^\lambda(\boldsymbol{k}) \, \boldsymbol{\epsilon}_j^\lambda(\boldsymbol{k}) = \delta_{ij} - \frac{\boldsymbol{k}_i \boldsymbol{k}_j}{\boldsymbol{k}^2} \qquad \text{(component form)}$

The two approaches to quantization are therefore equivalent.

$$\mathcal{L}_{\text{EM}} = \frac{1}{8\pi}\left(\partial_\mu \boldsymbol{A}\right)\cdot\partial^\mu \boldsymbol{A} \qquad \pi_j = \frac{1}{4\pi c^2}\dot{\boldsymbol{A}}_j$$

$\rightarrow$ $$\mathcal{H}_{\text{EM}} = \frac{1}{4\pi c^2}\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}} - \mathcal{L}_{\text{EM}}$$

$$= \frac{1}{8\pi}\left(\frac{1}{c^2}\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}} + \left(\partial_i \boldsymbol{A}_j\right)\partial_i \boldsymbol{A}_j\right)$$

In the Coulomb gauge,

$$\boldsymbol{E} = \frac{1}{c}\dot{\boldsymbol{A}}$$

Using $\partial_j \boldsymbol{A}_j = 0$,

$\rightarrow$ $$\left(\partial_j \boldsymbol{A}_k\right)\partial_k \boldsymbol{A}_j = \partial_j\left(\boldsymbol{A}_k \partial_k \boldsymbol{A}_j\right) = \partial_j \partial_k\left(\boldsymbol{A}_k \boldsymbol{A}_j\right) = 0$$

we have

$$\boldsymbol{B}^2 = \varepsilon_{ijk}\partial_j \boldsymbol{A}_k \varepsilon_{ilm}\partial_l \boldsymbol{A}_m = \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)\partial_j \boldsymbol{A}_k \partial_l \boldsymbol{A}_m$$

$$= \left(\partial_j \boldsymbol{A}_k\right)\partial_j \boldsymbol{A}_k - \left(\partial_j \boldsymbol{A}_k\right)\partial_k \boldsymbol{A}_j$$

$$= \left(\partial_j \boldsymbol{A}_k\right)\partial_j \boldsymbol{A}_k$$

$$= \partial_j\left(\boldsymbol{A}_k \partial_j \boldsymbol{A}_k\right) - \boldsymbol{A}_k \partial_j \partial_j \boldsymbol{A}_k$$

$$= \partial_j\left(\boldsymbol{A}_k \partial_j \boldsymbol{A}_k\right) - \boldsymbol{A}\cdot\nabla^2 \boldsymbol{A}$$

$\therefore$ $$\mathcal{H}_{\text{EM}} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

$$H_{\text{EM}} = \int d^3 r\, \mathcal{H}_{\text{EM}}$$

$$= \frac{1}{8\pi}\int d^3 r\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

$$= \frac{1}{8\pi}\int d^3 r\left(\frac{1}{c^2}\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}} - \boldsymbol{A}\cdot\nabla^2 \boldsymbol{A}\right)$$

Treated as the sum of 2 K-G fields, we have

$$H_{\text{EM}} = \int d^3 k \sum_{\lambda=1}^{2} \hbar\omega_{\boldsymbol{k}}\left(a_{\boldsymbol{k}}^{\lambda+} a_{\boldsymbol{k}}^{\lambda} + \frac{1}{2}\delta(0)\right)$$

where, since $m = 0$,

$$\omega_{\boldsymbol{k}} = c\,|\,\boldsymbol{k}\,|$$

Also,

$$\boldsymbol{p} = \int d^3 k \sum_{\lambda=1}^{2} \hbar\boldsymbol{k}\, a_{\boldsymbol{k}}^{\lambda+} a_{\boldsymbol{k}}^{\lambda}$$

$$\Theta^{0j} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial\, \partial_0 A^\lambda}\partial^j A^\lambda = c\,\pi_\lambda \partial^j A^\lambda$$

In the Coulomb gauge,

$$\Theta^{0j} = -\frac{1}{4\pi c}\dot{\boldsymbol{A}}_i \partial_j \boldsymbol{A}_i = \frac{1}{4\pi}\boldsymbol{E}\cdot\partial_j \boldsymbol{A}$$

$$(\boldsymbol{E}\times\boldsymbol{B})_i = [\boldsymbol{E}\times(\nabla\times\boldsymbol{A})]_i = \varepsilon_{ijk}\boldsymbol{E}_j \varepsilon_{klm}\partial_l \boldsymbol{A}_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\boldsymbol{E}_j \partial_l \boldsymbol{A}_m$$

$$= \boldsymbol{E}_j \partial_i \boldsymbol{A}_j - \boldsymbol{E}_j \partial_j \boldsymbol{A}_i$$

$$= \boldsymbol{E}\cdot\partial_i \boldsymbol{A} - \boldsymbol{E}\cdot\nabla \boldsymbol{A}_i$$

$$= \boldsymbol{E}\cdot\partial_i \boldsymbol{A} - \nabla(\boldsymbol{E}\,\boldsymbol{A}_i) + \boldsymbol{A}_i \nabla\cdot\boldsymbol{E}$$

$$= \boldsymbol{E} \cdot \partial_i \boldsymbol{A} - \nabla(\boldsymbol{E}\,\boldsymbol{A}_i)$$

$$\therefore \qquad \Theta^{0j} = \frac{1}{4\pi}\left[(\boldsymbol{E}\times\boldsymbol{B})_j + \nabla\left(\boldsymbol{E}\,\boldsymbol{A}_j\right)\right]$$

$$\frac{1}{c}\int d^3 r\, \Theta^{0j} = \frac{1}{4\pi c}\int d^3 r\, (\boldsymbol{E}\times\boldsymbol{B})_j = \int d^3 r\, \boldsymbol{g}_j$$

where $\boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B}$ is the EM momentum density.

We wish to show that

$$\boldsymbol{p} = \int d^3 r\, \boldsymbol{g} = \frac{1}{c}\int d^3 r\, \Theta^{0j}$$

using

$$\boldsymbol{A}(x) = \hbar c\sqrt{4\pi}\int \frac{d^3 k}{\sqrt{(2\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}}\sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^{\lambda}(\boldsymbol{k})\,\left(a_{\boldsymbol{k}}^{\lambda} e^{-i k\cdot x} + a_{\boldsymbol{k}}^{\lambda+} e^{i k\cdot x}\right)$$

The relevant procedure is well discussed in Chapter 3.

That $-\frac{1}{4\pi c^2}\int d^3 r\, \dot{\boldsymbol{A}}_i\, \partial_j \boldsymbol{A}_i$ should give something of the form

$$\int d^3 k \sum_{\lambda=1}^{2} f(\boldsymbol{k})\, a_{\boldsymbol{k}}^{\lambda+}\, a_{\boldsymbol{k}}^{\lambda}$$

is obvious. What needs to be done is to show that

$$f(\boldsymbol{k}) = \hbar\,\boldsymbol{k}$$

Thus,

$\dot{\boldsymbol{A}}_i$ gives a factor $\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}}(\mp i\,\omega_{\boldsymbol{k}})$

$\partial_j \boldsymbol{A}_i$ gives a factor $\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}}(\pm i\,\boldsymbol{k})$

$\int d^3 r$ gives a factor $(2\pi)^3\,\delta(\boldsymbol{k}-\boldsymbol{k}')$

where the upper / lower sign applies to the $a_{\boldsymbol{k}} / a_{\boldsymbol{k}}^{+}$ term.

Only the cross terms $a_{\boldsymbol{k}}^{+} a_{\boldsymbol{k}}$ or $a_{\boldsymbol{k}}\, a_{\boldsymbol{k}}^{+}$ survive so the over all factor is

$$-\frac{1}{4\pi c^2}\,\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}}\,(\mp i\,\omega_{\boldsymbol{k}})\,\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}}\,(\mp i\,\boldsymbol{k})\,(2\pi)^3$$

$$= \frac{1}{2}\hbar\,\boldsymbol{k}$$ as required.