5.2.b. Canonical Quantization : Vacuum, Lorentz Gauge

Coulomb gauge quantization is cumbersome (particularly for high spin theories) since Lorentz covariance is deliberately broken.

One way to preserve the Lorentz covariance is the Gupta-Bleuler scheme which starts with (see Kaku, §4.4)

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8\pi} \left(\partial_\mu A^\mu\right)^2$$

Thus, in the Lorentz gauge, $\partial_\mu A^\mu = 0$, $\mathcal{L} = \mathcal{L}_{\text{EM}}$.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\rightarrow \qquad \mathcal{L} = -\frac{1}{8\pi}\left[\partial_\mu A_\nu \cdot \partial^\mu A^\nu - \partial_\nu A_\mu \cdot \partial^\mu A^\nu + \left(\partial_\mu A^\mu\right)^2 \right]$$

$$\partial_\nu A_\mu \cdot \partial^\mu A^\nu = \partial_\nu \left(A_\mu \partial^\mu A^\nu \right) - A_\mu \partial^\mu \partial_\nu A^\nu$$

$$\left(\partial_\mu A^\mu\right)^2 = \partial_\mu A^\mu \cdot \partial_\nu A^\nu = \partial_\mu \left(A^\mu \partial_\nu A^\nu \right) - A^\mu \partial_\mu \partial_\nu A^\nu$$

$$= \partial_\mu \left(A^\mu \partial_\nu A^\nu \right) - A_\mu \partial^\mu \partial_\nu A^\nu$$

$$\rightarrow \qquad \mathcal{L} = -\frac{1}{8\pi} \partial_\mu A_\nu \cdot \partial^\mu A^\nu$$

where the divergence terms

$$\frac{1}{8\pi}\left[\partial_\nu \left(A_\mu \partial^\mu A^\nu \right) - \partial_\mu \left(A^\mu \partial_\nu A^\nu \right) \right]$$

were dropped.

$$\frac{\partial \mathcal{L}}{\partial\, \partial^\mu A^\nu} = -\frac{1}{4\pi} \partial_\mu A_\nu$$

Euler eq. :

$$-\frac{1}{4\pi} \partial^\mu \partial_\mu A_\nu = 0$$

which is what it should be in the Lorentz gauge.

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = \frac{\partial \mathcal{L}}{c\, \partial\, \partial^0 A^\mu} = -\frac{1}{4\pi c} \partial_0 A_\mu = -\frac{1}{4\pi c^2} \dot{A}_\mu$$

Covariant quantization:

$$\left[A_\mu(t, \boldsymbol{r}),\, A_\nu(t, \boldsymbol{r}') \right] = \left[\pi_\mu(t, \boldsymbol{r}),\, \pi_\nu(t, \boldsymbol{r}') \right] = 0$$

$$\left[A_\mu(t, \boldsymbol{r}),\, \pi^\nu(t, \boldsymbol{r}') \right] = i\hbar\, \delta_\mu^\nu\, \delta(\boldsymbol{r} - \boldsymbol{r}')$$

or $\qquad \left[A_\mu(t, \boldsymbol{r}),\, \pi_\nu(t, \boldsymbol{r}') \right] = i\hbar\, \eta_{\mu\nu}\, \delta(\boldsymbol{r} - \boldsymbol{r}')$

$$\left[A_\mu(t, \boldsymbol{r}),\, \dot{A}_\nu(t, \boldsymbol{r}') \right] = -4\pi c^2\, i\hbar\, \eta_{\mu\nu}\, \delta(\boldsymbol{r} - \boldsymbol{r}')$$

Plane wave expansion:

$$A_\mu(x) = \hbar c \sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3\, 2\hbar\omega_{\boldsymbol{k}}}}\, \epsilon_{\mu\lambda}(\boldsymbol{k}) \left(a_{\boldsymbol{k}}^\lambda e^{-i k\cdot x} + a_{\boldsymbol{k}}^{\lambda +} e^{i k \cdot x} \right)$$

$$A^\mu(x) = \hbar c \sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3\, 2\hbar\omega_{\boldsymbol{k}}}}\, \epsilon_\lambda^\mu(\boldsymbol{k}) \left(a_{\boldsymbol{k}}^\lambda e^{-i k\cdot x} + a_{\boldsymbol{k}}^{\lambda +} e^{i k \cdot x} \right)$$

$$\dot{A}_\mu(x) = i\hbar c\sqrt{4\pi}\int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\hbar\omega_{\boldsymbol{k}}}}\,\epsilon_{\mu\lambda}(\boldsymbol{k})\,\omega_{\boldsymbol{k}}\left(-a^{\lambda}_{\boldsymbol{k}}\,e^{-ik\cdot x} + a^{\lambda+}_{\boldsymbol{k}}\,e^{ik\cdot x}\right)$$

where $\{\epsilon_{\mu\lambda}(\boldsymbol{k})\}$ is a set of four 4-vectors when one of the indices is fixed.

Writing

$$\eta_{\mu\nu}\,\delta(\boldsymbol{r}-\boldsymbol{r}') = \eta_{\mu\nu}\int\frac{d^3k}{(2\pi)^3}\,e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}$$

we see that $[A_\mu(t,\boldsymbol{r}), \dot{A}_\nu(t,\boldsymbol{r})]$ can be reduced to such a form only if

1. $[a^{\lambda}_{\boldsymbol{k}}, a^{\lambda'}_{\boldsymbol{k}'}] = [a^{+\lambda}_{\boldsymbol{k}}, a^{+\lambda'}_{\boldsymbol{k}'}] = 0$,

   $[a^{\lambda}_{\boldsymbol{k}}, a^{+}_{\lambda'\boldsymbol{k}'}] \propto \delta^{\lambda}_{\lambda'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')$ or $[a^{\lambda}_{\boldsymbol{k}}, a^{+\lambda'}_{\boldsymbol{k}'}] \propto \eta^{\lambda\lambda'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')$

2. $\eta^{\lambda\lambda'}\,\epsilon_{\mu\lambda}(\boldsymbol{k})\,\epsilon^{\nu}_{\lambda'}(\boldsymbol{k}) = \epsilon^{\lambda}_{\mu}(\boldsymbol{k})\,\epsilon^{\nu}_{\lambda}(\boldsymbol{k}) = \delta^{\nu}_{\mu}$

   or $\eta^{\lambda\lambda'}\,\epsilon_{\mu\lambda}(\boldsymbol{k})\,\epsilon_{\nu\lambda'}(\boldsymbol{k}) = \epsilon_{\mu\lambda}(\boldsymbol{k})\,\epsilon^{\lambda}_{\nu}(\boldsymbol{k}) = \eta_{\mu\nu}$

The overall factor of $[a^{\lambda}_{\boldsymbol{k}}, a^{+\lambda'}_{\boldsymbol{k}'}]$ in $[A_\mu(t,\boldsymbol{r}), \dot{A}_\nu(t,\boldsymbol{r}')]$ is then

$$2\times\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}}\,\frac{i\hbar c\sqrt{4\pi}\;\omega_{\boldsymbol{k}}}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}} = i\hbar\,\frac{4\pi c^2}{(2\pi)^3}$$

Comparing this with

$$-4\pi c^2\,\frac{i\hbar\,\eta_{\mu\nu}}{(2\pi)^3}$$

we have $[a^{\lambda}_{\boldsymbol{k}}, a^{+\lambda'}_{\boldsymbol{k}'}] = -\eta^{\lambda\lambda'}\,\delta(\boldsymbol{k}-\boldsymbol{k}')$

$$\mathcal{H} = \pi_\mu\dot{A}^\mu - \mathcal{L} = -\frac{1}{4\pi c^2}\dot{A}_\mu\dot{A}^\mu + \frac{1}{8\pi}\partial_\mu A_\nu\cdot\partial^\mu A^\nu$$
$$= \frac{1}{8\pi}\left(-\frac{1}{c^2}\dot{A}_\mu\dot{A}^\mu + \nabla A_\mu\cdot\nabla A^\mu\right)$$

The overall factor of $a^{\lambda}_{\boldsymbol{k}}\,a^{\lambda'+}_{\boldsymbol{k}}\;/\;a^{\lambda+}_{\boldsymbol{k}}\,a^{\lambda'}_{\boldsymbol{k}}$ for the $\dot{A}_\mu\dot{A}^\mu$ term in $H = \int d^3r\,\mathcal{H}$ is

$$\left(-\frac{1}{8\pi c^2}\right)(2\pi)^3\left(\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}}\right)^2(\mp i\,\omega_{\boldsymbol{k}})(\pm i\,\omega_{\boldsymbol{k}})\,\eta_{\lambda\lambda'} = -\frac{1}{4}\hbar\,\omega_{\boldsymbol{k}}\,\eta_{\lambda\lambda'}$$

where we've used the completeness relation

$$\epsilon_{\mu\lambda}(\boldsymbol{k})\,\epsilon^{\mu\lambda'}(\boldsymbol{k}) = \delta^{\lambda'}_{\lambda} \qquad\text{or}\qquad \epsilon_{\mu\lambda}(\boldsymbol{k})\,\epsilon^{\mu}_{\lambda'}(\boldsymbol{k}) = \eta_{\lambda\lambda'}$$

$$\nabla A_\mu(x) = i\hbar c\sqrt{4\pi}\int\frac{d^3k}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}}\,\epsilon_{\mu\lambda}(\boldsymbol{k})\,\boldsymbol{k}\left(a^{\lambda}_{\boldsymbol{k}}\,e^{-ik\cdot x} - a^{\lambda+}_{\boldsymbol{k}}\,e^{ik\cdot x}\right)$$

The overall factor of $a^{\lambda}_{\boldsymbol{k}}\,a^{\lambda+}_{\boldsymbol{k}}\;/\;a^{\lambda+}_{\boldsymbol{k}}\,a^{\lambda}_{\boldsymbol{k}}$ for the $\nabla A_\mu\cdot\nabla A^\mu$ term in $H = \int d^3r\,\mathcal{H}$ is

$$\left(\frac{1}{8\pi}\right)(2\pi)^3\left(\frac{\hbar c\sqrt{4\pi}}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}}\right)^2(\pm i\,\boldsymbol{k})\cdot(\mp i\,\boldsymbol{k})\,\eta_{\lambda\lambda'} = -\frac{1}{4}\,\frac{\hbar\boldsymbol{k}^2}{\omega_{\boldsymbol{k}}}\,\eta_{\lambda\lambda'}$$

Hence,

$$H = -\int d^3k\,\frac{1}{4}\left(\hbar\,\omega_{\boldsymbol{k}} + \frac{\hbar\boldsymbol{k}^2}{\omega_{\boldsymbol{k}}}\right)\eta_{\lambda\lambda'}\left(a^{\lambda}_{\boldsymbol{k}}\,a^{\lambda'+}_{\boldsymbol{k}} + a^{\lambda+}_{\boldsymbol{k}}\,a^{\lambda'}_{\boldsymbol{k}}\right)$$

$$= -\int d^3k \, \frac{1}{2} \hbar \omega_{\boldsymbol{k}} \left[2 \eta_{\lambda\lambda'} a_{\boldsymbol{k}}^{\lambda +} a_{\boldsymbol{k}}^{\lambda'} - \delta(0) \right]$$

$$= \int d^3k \; \hbar \omega_{\boldsymbol{k}} \left[a_{\boldsymbol{k}}^{i+} a_{\boldsymbol{k}}^{i} - a_{\boldsymbol{k}}^{0+} a_{\boldsymbol{k}}^{0} + \frac{1}{2} \delta(0) \right]$$

Since we've assumed all 4 A^{μ}'s to be independent, the Lorentz condition $\partial^{\mu} A_{\mu} = 0$ must be imposed on the states. Since

$$\partial^{\mu} A_{\mu}(x) = \hbar c \sqrt{4\pi} \int \frac{d^3k}{\sqrt{(2\pi)^3 \, 2 \hbar\omega_{\boldsymbol{k}}}} \, i k^{\mu} \epsilon_{\mu\lambda}(\boldsymbol{k}) \left(-a_{\boldsymbol{k}}^{\lambda} e^{-i k\cdot x} + a_{\boldsymbol{k}}^{\lambda +} e^{i k \cdot x} \right)$$

However, if we set

$$\partial^{\mu} A_{\mu}(x) \left| \text{phys} \right\rangle = 0$$

there is no non-trivial solutions.

The usual way out is to demand instead

$$\partial^{\mu} A_{\mu}^{(+)}(x) \left| \text{phys} \right\rangle = 0$$

where (+) denotes the positive energy part, i.e.,

$$\partial^{\mu} A_{\mu}^{(+)}(x) = -\hbar c \sqrt{4\pi} \int \frac{d^3k}{\sqrt{(2\pi)^3 \, 2 \hbar\omega_{\boldsymbol{k}}}} \, i k^{\mu} \epsilon_{\mu\lambda}(\boldsymbol{k}) \, a_{\boldsymbol{k}}^{\lambda} e^{-i k\cdot x}$$

Which means

$$k^{\mu} \epsilon_{\mu\lambda}(\boldsymbol{k}) \, a_{\boldsymbol{k}}^{\lambda} \left| \text{phys} \right\rangle = 0 \qquad\qquad \forall \; k^{\mu} = (\, |\boldsymbol{k}|, \, \boldsymbol{k} \,)$$

A solution to this is to set

$$k^{\mu} \epsilon_{\mu 1}(\boldsymbol{k}) = k^{\mu} \epsilon_{\mu 2}(\boldsymbol{k}) = 0 \qquad\qquad (\text{ transverse to } k^{\mu} \,)$$

and $\quad k^{\mu} \epsilon_{\mu 0}(\boldsymbol{k}) = k^{\mu} \epsilon_{\mu 3}(\boldsymbol{k})$

That this is possible can be demonstrated in a frame where

$$k^{\mu} = (\, |\boldsymbol{k}|, 0, 0, \, |\boldsymbol{k}| \,)$$

so that a solution is

$$\epsilon_{\mu 0} = (1, 0, 0, 0)$$

$$\epsilon_{\mu 1} = (0, 1, 0, 0)$$

$$\epsilon_{\mu 2} = (0, 0, 1, 0)$$

$$\epsilon_{\mu 3} = (0, 0, 0, 1)$$

For this choice,

$$k^{\mu} \epsilon_{\mu\lambda}(\boldsymbol{k}) \, a_{\boldsymbol{k}}^{\lambda} \left| \text{phys} \right\rangle = |\boldsymbol{k}| \left(a_{\boldsymbol{k}}^{0} - a_{\boldsymbol{k}}^{3} \right) \left| \text{phys} \right\rangle = 0$$

→ $\quad \left(a_{\boldsymbol{k}}^{0} - a_{\boldsymbol{k}}^{3} \right) \left| \text{phys} \right\rangle = 0$

or $\quad a_{\boldsymbol{k}}^{0} \left| \text{phys} \right\rangle = a_{\boldsymbol{k}}^{3} \left| \text{phys} \right\rangle$

Thus,

$$\left\langle \text{phys}' \right| a_{\boldsymbol{k}}^{0+} a_{\boldsymbol{k}}^{0} - a_{\boldsymbol{k}}^{3+} a_{\boldsymbol{k}}^{3} \left| \text{phys} \right\rangle = 0$$

→ $\quad \langle \text{phys}' \,|\, H_{\boldsymbol{k}} \,|\, \text{phys} \rangle = \left\langle \text{phys}' \right| \hbar \omega_{\boldsymbol{k}} \left[a_{\boldsymbol{k}}^{1+} a_{\boldsymbol{k}}^{1} + a_{\boldsymbol{k}}^{2+} a_{\boldsymbol{k}}^{2} + \frac{1}{2} \delta(0) \right] \left| \text{phys} \right\rangle$

which is the same as the Coulomb gauge.