# 5.2.c. Canonical Quantization : Linear Medium, Coulomb Gauge

## Basics

From 5.1._MaxwellEquations.pdf, we have, in general,

$$\mathcal{L}_{EM} = = -\frac{1}{8\pi\mu_m}\left(\varepsilon\mu_m\partial_0 A_i\cdot\partial^0 A^i + \partial_j A_i\cdot\partial^j A^i\right)$$

$$-\frac{1}{8\pi}\left(\varepsilon\,\partial_i A_0\cdot\partial^i A^0 + \frac{1}{\mu_m}A_j\,\partial^j\partial_i A^i\right)$$

$$-\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) = \frac{4\pi\rho_f}{\varepsilon}$$

$$\frac{\varepsilon\mu_m}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} + \nabla\left(\frac{\varepsilon\mu_m}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\boldsymbol{A}\right) = \frac{4\pi}{c}\mu_m\boldsymbol{J}_f$$

$$\Theta^{0j} = \frac{\partial\mathcal{L}_{EM}}{\partial\,\partial_0 A^\lambda}\partial^j A^\lambda = c\,\pi_\lambda\,\partial^j A^\lambda$$

## Coulomb Gauge in Linear Medium

$$J_f^\mu = 0$$

$$A^0 = \varphi = 0 \qquad \nabla\cdot\boldsymbol{A} = \partial_i A^i = 0$$

$\rightarrow$ $\nabla^2\varphi = 0$( automatically satisfied )

$$\frac{\varepsilon\mu_m}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} = 0$$

$$\mathcal{L}_{EM} = -\frac{1}{8\pi\mu_m}\left(\varepsilon\mu_m\partial_0 A_i\cdot\partial^0 A^i + \partial_j A_i\cdot\partial^j A^i\right)$$

$$= \frac{1}{8\pi\mu_m}\left[\varepsilon\mu_m(\partial_0\boldsymbol{A}_i)\,\partial^0\boldsymbol{A}_i + (\partial_j\boldsymbol{A}_i)\,\partial^j\boldsymbol{A}_i\right]$$

Comparing with the real Klein-Gordon field where

$$\mathcal{L} = \frac{1}{2}f\left(\partial_\mu\phi\cdot\partial^\mu\phi - \frac{m^2c^2}{\hbar^2}\phi^2\right)$$

we see that each $A^i = \boldsymbol{A}_i$ can be treated as a massless K-G field with

$$\partial_0 = \sqrt{\varepsilon\mu_m}\,\frac{\partial}{c\,\partial t} \qquad \& \qquad f = \frac{1}{4\pi\mu_m}$$

i.e., the speed of light is reduced to $\frac{c}{\sqrt{\varepsilon\mu_m}}$.

Everything from here onward is analogous to the vacuum case so we need only list the salient points.

Comparing with

$$\phi(x) = \frac{\hbar c}{\sqrt{f}}\int\frac{d^3k}{\sqrt{(2\pi)^3\,2\hbar\omega_{\boldsymbol{k}}}}\left(a_{\boldsymbol{k}}\,e^{-ik\cdot x} + a_{\boldsymbol{k}}^+\,e^{ik\cdot x}\right)$$

we have

$$\frac{c}{\sqrt{f}} = \frac{c}{\sqrt{\varepsilon\,\mu_m}} \sqrt{4\,\pi\,\mu_m} = 2\,c\sqrt{\frac{4\,\pi}{\varepsilon}}$$

$$\boldsymbol{A}(x) = \hbar\,c\sqrt{\frac{4\,\pi}{\varepsilon}} \int \frac{d^3 k}{\sqrt{(2\,\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}} \left( \boldsymbol{a}_{\boldsymbol{k}}\, e^{-i\,k\cdot x} + \boldsymbol{a}_{\boldsymbol{k}}^{+}\, e^{i\,k\cdot x} \right)$$

where

$$\omega_{\boldsymbol{k}} = \frac{c}{\sqrt{\varepsilon\,\mu_m}} \left| \boldsymbol{k} \right|$$

$\nabla\cdot\boldsymbol{A} = 0$

$$\rightarrow \qquad \boldsymbol{A}(x) = \hbar\,c\sqrt{\frac{4\,\pi}{\varepsilon}} \int \frac{d^3 k}{\sqrt{(2\,\pi)^3\, 2\,\hbar\omega_{\boldsymbol{k}}}} \sum_{\lambda=1}^{2} \boldsymbol{\epsilon}^{\lambda}(\boldsymbol{k}) \left( a_{\boldsymbol{k}}^{\lambda}\, e^{-i\,k\cdot x} + a_{\boldsymbol{k}}^{\lambda+}\, e^{i\,k\cdot x} \right)$$

where

$$\boldsymbol{k}\cdot\boldsymbol{\epsilon}^{\lambda}(\boldsymbol{k}) = 0 \qquad \forall\ \lambda = 1, 2$$

$$\& \qquad \boldsymbol{\epsilon}^{\lambda}(\boldsymbol{k})\cdot\boldsymbol{\epsilon}^{\lambda'}(\boldsymbol{k}) = \delta^{\lambda\lambda'}$$

To quantize the EM field, one simply sets

$$\left[ a_{\boldsymbol{k}}^{\lambda},\, a_{\boldsymbol{k'}}^{+\lambda'} \right] = \delta^{\lambda\lambda'}\,\delta(\boldsymbol{k} - \boldsymbol{k}')$$

while

$$\left[ a_{\boldsymbol{k}}^{\lambda},\, a_{\boldsymbol{k'}}^{\lambda'} \right] = \left[ a_{\boldsymbol{k}}^{+\lambda},\, a_{\boldsymbol{k'}}^{+\lambda'} \right] = 0$$

Canonical quantization starts with

$$\pi_j = \frac{\partial\,\mathcal{L}}{\partial\,\dot{A}^j} = \frac{\partial\,\mathcal{L}}{c\,\partial\,\partial^0 A^j} = -\frac{1}{4\,\pi}\,\varepsilon\,F_{0\,j} = -\frac{1}{4\,\pi}\,\varepsilon\,\boldsymbol{E}_j$$

$$= -\frac{1}{4\,\pi\,c}\,\varepsilon\,\partial_0 A_j = \frac{1}{4\,\pi\,c^2}\,\varepsilon\,\dot{\boldsymbol{A}}_j$$

Analogous to the vacuum case, we have

$$\left[ \boldsymbol{A}_i(t, \boldsymbol{r}),\ \boldsymbol{\pi}_j(t, \boldsymbol{r}') \right] = i\,\hbar\,\overline{\delta}_{i\,j}(\boldsymbol{r} - \boldsymbol{r}')$$

$$\text{or} \qquad \left[ \boldsymbol{A}_i(t, \boldsymbol{r}),\ \dot{\boldsymbol{A}}_j(t, \boldsymbol{r}') \right] = \frac{4\,\pi\,c^2}{\varepsilon}\, i\,\hbar\,\overline{\delta}_{i\,j}(\boldsymbol{r} - \boldsymbol{r}')$$

where

$$\overline{\delta}_{i\,j}(\boldsymbol{r} - \boldsymbol{r}') = \int \frac{d^3 k}{(2\,\pi)^3}\, e^{i\,\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \left( \delta_{i\,j} - \frac{\boldsymbol{k}_i\,\boldsymbol{k}_j}{\boldsymbol{k}^2} \right)$$

$$= \left( \delta_{i\,j} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(\boldsymbol{r} - \boldsymbol{r}')$$

$$\mathcal{L}_{\text{EM}} = \frac{1}{8\,\pi\,\mu_m} \left[ \varepsilon\,\mu_m\,(\partial_0 \boldsymbol{A}_i)\,\partial^0 \boldsymbol{A}_i + \left(\partial_j \boldsymbol{A}_i\right) \partial^j \boldsymbol{A}_i \right]$$

$$= \frac{1}{8\,\pi} \left[ \varepsilon\,(\partial_0 \boldsymbol{A})\cdot\partial^0 \boldsymbol{A} + \frac{1}{\mu_m} \left(\partial_j \boldsymbol{A}\right)\cdot\partial^j \boldsymbol{A} \right]$$

$$= \frac{1}{8\,\pi} \left( \frac{1}{c^2}\,\varepsilon\,\dot{\boldsymbol{A}}\cdot\dot{\boldsymbol{A}} - \frac{1}{\mu_m} \left(\partial_j \boldsymbol{A}\right)\cdot\partial_j \boldsymbol{A} \right)$$

$$= \frac{1}{8\,\pi} \left( \varepsilon\,\boldsymbol{E}^2 - \frac{1}{\mu_m}\,\boldsymbol{B}^2 \right)$$

$$\pi_j = \frac{1}{4\pi c^2} \varepsilon \dot{A}_j$$

$$\to \qquad \mathcal{H}_{EM} = \frac{1}{4\pi c^2} \varepsilon \dot{\boldsymbol{A}} \cdot \dot{\boldsymbol{A}} - \mathcal{L}_{EM}$$

$$= \frac{1}{8\pi} \left( \frac{1}{c^2} \varepsilon \dot{\boldsymbol{A}} \cdot \dot{\boldsymbol{A}} + \frac{1}{\mu_m} (\partial_j \boldsymbol{A}) \cdot \partial_j \boldsymbol{A} \right)$$

$$= \frac{1}{8\pi} \left( \varepsilon \boldsymbol{E}^2 + \frac{1}{\mu_m} \boldsymbol{B}^2 \right)$$

$$H_{EM} = \int d^3 r \, \mathcal{H}_{EM}$$

$$= \frac{1}{8\pi} \int d^3 r \left( \varepsilon \boldsymbol{E}^2 + \frac{1}{\mu_m} \boldsymbol{B}^2 \right)$$

$$= \frac{1}{8\pi} \int d^3 r \left( \frac{1}{c^2} \varepsilon \dot{\boldsymbol{A}} \cdot \dot{\boldsymbol{A}} - \frac{1}{\mu_m} \boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} \right)$$

Treated as the sum of 2 K-G fields with $c \to \frac{c}{\sqrt{\varepsilon \mu_m}}$, we have

$$H_{EM} = \int d^3 k \sum_{\lambda=1}^{2} \hbar \omega_{\boldsymbol{k}} \left( a_{\boldsymbol{k}}^{\lambda +} a_{\boldsymbol{k}}^{\lambda} + \frac{1}{2} \delta(0) \right)$$

Also,

$$\boldsymbol{p} = \int d^3 k \sum_{\lambda=1}^{2} \hbar \boldsymbol{k} \, a_{\boldsymbol{k}}^{\lambda +} a_{\boldsymbol{k}}^{\lambda}$$

$$\Theta^{0j} = -\frac{\varepsilon}{4\pi c} \dot{\boldsymbol{A}}_i \partial_j \boldsymbol{A}_i = \frac{\varepsilon}{4\pi} \boldsymbol{E} \cdot \partial_j \boldsymbol{A}$$

$$= \frac{\varepsilon}{4\pi} \left[ (\boldsymbol{E} \times \boldsymbol{B})_j + \nabla (\boldsymbol{E} \, \boldsymbol{A}_j) \right]$$

$$\frac{1}{c} \int d^3 r \, \Theta^{0j} = \frac{\varepsilon}{4\pi c} \int d^3 r \, (\boldsymbol{E} \times \boldsymbol{B})_j = \int d^3 r \, \boldsymbol{g}_j$$

where $\boldsymbol{g} = \frac{\varepsilon}{4\pi c} \boldsymbol{E} \times \boldsymbol{B}$ is the EM momentum density.