5.3. Interaction with Matter Field

ref: M.Kaku,"Quantum Field Theory", §4.2.

In classical mechanics, the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)$$

can be derived by applying the minimal coupling principle

$$p^\mu \quad\to\quad p^\mu - \frac{q}{c}A^\mu$$

to the Hamiltonian of a free particle (see H.Goldstein, "Classical Mechanics", 2nd ed., §§1.5, 8.2).

The principle is also used in quantum theories to introduce the interaction between matter & EM fields.

Note that in quantum theories, the principle can take the form

$$\partial_\mu \quad\to\quad \partial_\mu + i\frac{q}{\hbar c}A_\mu$$

It turns out this principle is just the consequence of promoting a global gauge symmetry of the system to a local one.

The matter Lagrangians of complex fields (Schrodinger, K-G, Dirac, ...) we studied so far are all invariant under a global gauge (or phase) transformation

$$\phi(x) \quad\to\quad e^{i\Lambda}\phi(x) \qquad\qquad \Lambda = \text{constant}$$

since ϕ always appear in $\mathcal{L}$ in the invariant forms $f(\phi^+\phi)$, $\partial_\mu\phi^+\cdot\partial^\nu\phi$, $\phi^+\partial^\nu\phi$, ...

When this is promoted to a local gauge transformation, i.e.,

$$\phi(x) \quad\to\quad e^{i\Lambda(x)}\phi(x)$$

term $f(\phi^+\phi)$ remains invariant, but

$$\partial_\mu\phi \quad\to\quad e^{i\Lambda}\left(\partial_\mu\phi + i\phi\,\partial_\mu\Lambda\right)$$

so that, say,

$$\phi^+\partial_\mu\phi \quad\to\quad \phi^+\partial_\mu\phi + i\phi^+\phi\,\partial_\mu\Lambda$$

is not invariant.

On the other hand, if

$$A_\mu \quad\to\quad A_\mu - \frac{1}{\alpha}\partial_\mu\Lambda$$

where α is any real constant, then

$$\left(\partial_\mu + i\alpha A_\mu\right)\phi \quad\to\quad e^{i\Lambda}\left[\partial_\mu\phi + i\phi\,\partial_\mu\Lambda + i\alpha\left(A_\mu - \frac{1}{\alpha}\partial_\mu\Lambda\right)\phi\right]$$

$$= e^{i\Lambda}\left(\partial_\mu + i\alpha A_\mu\right)\phi$$

Define the covariant derviative as

$$D_\mu = \partial_\mu + i\alpha A_\mu$$

$$D_\mu^+ = \partial_\mu - i\alpha A_\mu \quad \text{since} \quad A^+ = A$$

then

$$D_\mu\phi \quad\to\quad e^{i\Lambda}D_\mu\phi$$

$$\left(D_\mu\phi\right)^+ = D_\mu^+\phi^+ \;\to\; e^{-i\Lambda}D_\mu^+\phi^+ \qquad\qquad (A\ \&\ \phi \text{ commute })$$

so that all the derivative terms mentioned above are invariant if we replace ∂_μ with D_μ, i.e., $\partial_\mu\phi \to D_\mu\phi$, $\partial_\mu\phi^+ \to \left(D_\mu\phi\right)^+$.

The minimal coupling principle is recovered if we identify A_μ with the EM vector potential and set $\alpha = \frac{q}{\hbar c}$ so that

$$D_\mu = \partial_\mu + i \frac{q}{\hbar c} A_\mu$$

This is acceptable since the local gauge transformation

$$A_\mu \quad \to \quad A_\mu - \frac{1}{\alpha} \partial_\mu \Lambda$$

is just an EM gauge transformation

$$A_\mu \quad \to \quad A_\mu + \partial_\mu f$$

with $f = -\frac{\hbar c}{q} \Lambda$.

Complex Klein-Gordon Field

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{EM}}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{\text{matter}} = f\left[\left(\partial_\mu \phi^+\right) \partial^\mu \phi - V\left(\phi^+ \phi\right) \right]$$

Minimal coupling

$$\to \qquad \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} = f\left[\left(D_\mu \phi\right)^+ D^\mu \phi - V\left(\phi^+ \phi\right) \right]$$

$$= f\left[\left(\partial_\mu - i \frac{q}{\hbar c} A_\mu\right) \phi^+ \cdot \left(\partial^\mu + i \frac{q}{\hbar c} A^\mu\right) \phi - V\left(\phi^+ \phi\right) \right]$$

Euler eq. for A^μ :

$$\frac{\partial \mathcal{L}}{\partial \partial^\mu A^\nu} = \frac{\partial \mathcal{L}_{\text{EM}}}{\partial \partial^\mu A^\nu} = -\frac{1}{4\pi} F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = \frac{\partial \mathcal{L}_{\text{int}}}{\partial A^\nu}$$

$$= i \frac{q}{\hbar c} f\left[-\phi^+ D_\nu \phi + \left(D_\nu \phi\right)^+ \phi \right]$$

$$= i \frac{q}{\hbar c} f\left[-\phi^+ \left(\partial_\nu + i \frac{q}{\hbar c} A_\nu\right) \phi + \left(\partial_\nu - i \frac{q}{\hbar c} A_\nu\right) \phi^+ \cdot \phi \right]$$

$$\to \qquad \partial^\mu F_{\mu\nu} = i \frac{4\pi q}{\hbar c} f\left[-\phi^+ D_\nu \phi + \left(D_\nu \phi\right)^+ \phi \right]$$

Comparing with the Maxwell eq.

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

we have

$$J_\nu = i \frac{q}{\hbar} f\left[\left(D_\nu \phi\right)^+ \phi - \phi^+ D_\nu \phi \right]$$