5.4. Anderson-Higgs Mechanism

K-G field with EM field in vacuum:

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{EM}}$$

$$\mathcal{L}_{\text{EM}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{int}} = f\left[\left(\partial^\mu - i\frac{q}{\hbar c} A^\mu\right)\phi^+ \cdot \left(\partial_\mu + i\frac{q}{\hbar c} A_\mu\right)\phi - V(\phi^+\phi)\right]$$

$$V(\phi^+\phi) = -\gamma\phi^+\phi + \frac{g}{2}(\phi^+\phi)^2 \quad \text{with} \quad \gamma,\ g > 0$$

Classical vacuum (minimizes classical $\mathcal{H}$) :

$$\phi(x) = v = \sqrt{\frac{\gamma}{g}} \qquad A_\mu(x) = 0$$

Infinitely degenerate due to gauge symmetry.

Consider a low excited state accompanied by a gauge transformation

$$\phi(x) = e^{i\chi(x)}[v + \eta(x)]$$

$$A_\mu(x) = U_\mu(x) - \frac{\hbar c}{q}\partial_\mu\chi(x)$$

where η & U are real and $U^+ = U$.

$$\rightarrow \quad \left(\partial_\mu + i\frac{q}{\hbar c}A_\mu\right)\phi = e^{i\chi}\left[i(v+\eta)\left(\partial_\mu\chi + \frac{q}{\hbar c}A_\mu\right) + \partial_\mu\eta\right]$$

$$= e^{i\chi}\left[i\frac{q}{\hbar c}(v+\eta)U_\mu + \partial_\mu\eta\right]$$

$$\left(\partial^\mu - i\frac{q}{\hbar c}A^\mu\right)\phi^+ = e^{-i\chi}\left[-i\frac{q}{\hbar c}(v+\eta)U^\mu + \partial^\mu\eta\right]$$

$$\therefore \quad \left(\partial_\mu - i\frac{q}{\hbar c}A_\mu\right)\phi^+\left(\partial_\mu + i\frac{q}{\hbar c}A_\mu\right)\phi$$

$$= \left[\partial^\mu\eta - i\frac{q}{\hbar c}(v+\eta)U^\mu\right]\left[\partial_\mu\eta + i\frac{q}{\hbar c}(v+\eta)U_\mu\right]$$

$$= \partial^\mu\eta\cdot\partial_\mu\eta + \left(\frac{q}{\hbar c}\right)^2(v+\eta)^2 U^\mu U_\mu$$

$$= \partial^\mu\eta\cdot\partial_\mu\eta + \left(\frac{qv}{\hbar c}\right)^2 U^\mu U_\mu + \ldots$$

$$V(\phi^+\phi) = -\gamma(v+\eta)^2 + \frac{g}{2}(v+\eta)^4$$

$$= -\gamma(v^2 + 2v\eta + \eta^2) + \frac{g}{2}(v^4 + 4v^3\eta + 6v^2\eta^2) + \ldots$$

$$= \gamma\left(-\frac{1}{2}v^2 + 2\eta^2\right) + \ldots \qquad \left(v^2 = \frac{\gamma}{g}\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \partial_\mu U_\nu - \frac{\hbar c}{q}\partial_\mu\partial_\nu\chi - \partial_\nu U_\mu + \frac{\hbar c}{q}\partial_\nu\partial_\mu\chi$$

$$= \partial_\mu U_\nu - \partial_\nu U_\mu \equiv G_{\mu\nu}$$

Thus, to 2nd order in the fields,

$$\mathcal{L} \simeq \mathcal{L}_2 = -\frac{1}{16\pi} G_{\mu\nu} G^{\mu\nu} + f\left[\partial^\mu \eta \cdot \partial_\mu \eta - 2\gamma \eta^2 + \left(\frac{q v}{\hbar c}\right)^2 U^\mu U_\mu\right]$$

where a constant term $\frac{1}{2} f \gamma v^2$ is dropped.

Lorentz gauge:

$$0 = \partial^\mu A_\mu = \partial^\mu U_\mu(x) - \frac{\hbar c}{q} \partial^\mu \partial_\mu \chi(x)$$

Since the Goldstone mode χ doesn't appear in the quadratic $\mathcal{L}_2$, we can set $\partial^\mu \partial_\mu \chi = 0$ and get

$$\partial^\mu U_\mu(x) = 0 \qquad \text{(Lorentz condition)}$$

$$\frac{\partial \mathcal{L}_2}{\partial \partial^\mu U^\nu} = -\frac{1}{4\pi} G_{\mu\nu}$$

$$\frac{\partial \mathcal{L}_2}{\partial U^\nu} = 2 f \left(\frac{q v}{\hbar c}\right)^2 U_\nu$$

$$\frac{\partial \mathcal{L}_2}{\partial \partial^\mu \eta} = 2 f \partial_\mu \eta$$

$$\frac{\partial \mathcal{L}_2}{\partial \eta} = -4 f \gamma \eta$$

Euler eqs.:

$$\partial^\mu \partial_\mu \eta + 2\gamma \eta = 0$$

→ η is a K-G field with mass $m_\eta = \frac{\hbar}{c}\sqrt{2\gamma}$.

The coherence length for the boson condensation (see 4.2._RealKlein-GordonField.pdf) is

$$\xi = \frac{\hbar}{m_\eta c} = \frac{1}{\sqrt{2\gamma}}$$

$$\partial_\mu G^{\mu\nu} = -8\pi f \left(\frac{q v}{\hbar c}\right)^2 U^\nu$$

$$= \partial_\mu\left(\partial^\mu U^\nu - \partial^\nu U^\mu\right) = \partial_\mu \partial^\mu U^\nu - \partial^\nu \partial_\mu U^\mu = \partial_\mu \partial^\mu U^\nu$$

→ U^ν is a vector K-G field with mass

$$m_U = \frac{\hbar}{c}\sqrt{8\pi f \left(\frac{q v}{\hbar c}\right)^2} = \frac{q v}{c^2}\sqrt{8\pi f}$$

The penetration depth (decay length of the massive photon) is

$$\lambda = \frac{\hbar}{m_U c} = \frac{\hbar c}{q v \sqrt{8\pi f}}$$

Interpretation

When bose condensation doesn't occur ($\gamma < 0$), there's no broken symmetry so A_μ remains massless, while ϕ can be treated as 2 real massive scalar fields. Furthermore, the vector field A_μ contains only 2 independent (transverse) components.

When bose condensation does occur ($\gamma > 0$), the broken symmetry makes A_μ or U_μ massive, while ϕ is represented by only 1 massive scalar field η.

Since the total degrees of freedom of the system should not change, we expect 3 independent components in U_μ.

Note that gauge symmetry is not broken. The 3rd independent (longitudinal) component in the gauge field U_μ comes from absorbing the massless Goldstone mode χ. This is called the Anderson-Higgs mechanism.