5.5. Massive Vector Field

From §5.4, the free field Lagrangian for U is

$$\mathcal{L} = -\frac{1}{16\pi} G_{\mu\nu} G^{\mu\nu} + \frac{1}{8\pi}\left(\frac{m_U c}{\hbar}\right)^2 U^\mu U_\mu$$

where

$$G_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \qquad\qquad \frac{1}{8\pi}\left(\frac{m_U c}{\hbar}\right)^2 = f\left(\frac{q v}{\hbar c}\right)^2$$

The Euler eq. is

$$\partial_\mu G^{\mu\nu} = -\left(\frac{m_U c}{\hbar}\right)^2 U^\nu$$
$$= \partial_\mu\left(\partial^\mu U^\nu - \partial^\nu U^\mu\right)$$

$\therefore$ $\mathcal{L}$ describes real K-G bosons with mass m_U.

Due to the mass term, $\mathcal{L}$ is not invariant under the gauge transformation

$$U_\mu(x) \;\rightarrow\; U_\mu(x) - \frac{\hbar c}{q}\partial_\mu \chi(x)$$

where χ comes from $\phi \rightarrow e^{i\chi}\phi$.

Thus, $\mathcal{L}$ does not describe a gauge theory.

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{U}^\mu} = -\frac{1}{4\pi c} G_{0\mu}$$

$\rightarrow$ $\pi_0 = 0$

Setting $\nu = 0$ in the Euler eq., we have

$$\partial^\mu G_{\mu 0} = -\left(\frac{m_U c}{\hbar}\right)^2 U_0$$
$$= -4\pi c\, \partial^\mu \pi_\mu = -4\pi c\, \partial^k \pi_k$$

Thus

$$U_0 = \frac{4\pi}{c}\left(\frac{\hbar}{m_U}\right)^2 \partial^k \pi_k$$

i.e., U_0 is not an independent variable.

If we simply set $U_0 = 0$, which is akin to the Coulomb gauge, we have

$$\left[U_j(t,\mathbf{r}),\; U_k(t,\mathbf{r}')\right] = \left[\pi_j(t,\mathbf{r}),\; \pi_k(t,\mathbf{r}')\right] = 0$$
$$\left[U_j(t,\mathbf{r}),\; \pi_k(t,\mathbf{r}')\right] = i\hbar\,\delta_{jk}\,\delta(\mathbf{r}-\mathbf{r}')$$
$$\left[U_j(t,\mathbf{r}),\; \dot{U}_k(t,\mathbf{r}')\right] = -4\pi c^2 i\hbar\,\delta_{jk}\,\delta(\mathbf{r}-\mathbf{r}')$$

Lorentz gauge

In the Lorentz gauge (see 5.2.b._CanonicalQuantization_VacuumLorentzGauge.pdf)

$$\mathcal{L} = -\frac{1}{8\pi}\left[\partial_\mu U_\nu \cdot \partial^\mu U^\nu - \left(\frac{m_U c}{\hbar}\right)^2 U^\mu U_\mu\right]$$
$$= -\frac{1}{8\pi}\sum_{\nu=0}^{3} \eta^{\alpha\alpha}\left[\partial_\mu U_\alpha \cdot \partial^\mu U_\alpha - \left(\frac{m_U c}{\hbar}\right)^2 U_\alpha^2\right]$$

after dropping some divergence terms. Hence $\mathcal{L}$ is a sum of 4 real K-G fields with coefficients $\eta^{\alpha\alpha}$.

The Euler eq. becomes the K-G eqs:

$$\partial_\mu \partial^\mu U^\nu - \left(\frac{m_U c}{\hbar}\right)^2 U^\nu = 0 \qquad \text{(Proca eq.)}$$

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{U}^\mu} = -\frac{1}{4\pi c^2} \dot{U}_\mu$$

Also, from the Lorentz condition,

$$\partial^\mu U_\mu = 0$$

we get

$$\frac{1}{c} \dot{U}_0 = -\partial^k U_k = \partial_k U_k$$

As in the general case, U_0 is not an independent variable.

Or, we can generalize the method used in 5.2.a._CanonicalQuantization_VacuumCoulombGauge.pdf .
to write it in the covariant form

$$\left[U_\mu(t, \boldsymbol{r}) ,\ \dot{U}^\nu(t, \boldsymbol{r}') \right] = -4\pi c^2 i \hbar \overline{\delta}_\mu^\nu(\boldsymbol{r} - \boldsymbol{r}')$$

where

$$\overline{\delta}_\mu^\nu(\boldsymbol{r} - \boldsymbol{r}') = \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')} \left(\delta_\mu^\nu - \frac{k_\mu k^\nu}{k^2} \right)$$

$$= \left[\delta_\mu^\nu + \left(\frac{\hbar}{m_U c}\right)^2 \partial_\mu \partial^\nu \right] \int \frac{d^3 k}{(2\pi)^3} e^{i \boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}')}$$

$$= \left[\delta_\mu^\nu + \left(\frac{\hbar}{m_U c}\right)^2 \partial_\mu \partial^\nu \right] \delta(\boldsymbol{r} - \boldsymbol{r}')$$

where $\qquad k^2 = k_\mu k^\mu = \left(\frac{m_U c}{\hbar}\right)^2$

Thus,

$$\left[U_\mu(t, \boldsymbol{r}) ,\ \dot{U}_\nu(t, \boldsymbol{r}') \right] = -4\pi c^2 i \hbar \overline{\delta}_{\mu\nu}(\boldsymbol{r} - \boldsymbol{r}')$$

$$\overline{\delta}_{\mu\nu}(\boldsymbol{r} - \boldsymbol{r}') = \left[\eta_{\mu\nu} + \left(\frac{\hbar}{m_U c}\right)^2 \partial_\mu \partial_\nu \right] \delta(\boldsymbol{r} - \boldsymbol{r}')$$

In order to compare with

$$\phi(x) = \frac{\hbar c}{\sqrt{f}} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2 \hbar \omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k}} e^{-i k \cdot x} + a_{\boldsymbol{k}}^+ e^{i k \cdot x} \right)$$

to get a plane wave expansion of U_μ , we set $f = \frac{1}{4\pi}$ instead of $f = \frac{1}{8\pi}$ because the U_μ are real fields. Hence,

$$U_\mu(x) = \hbar c \sqrt{4\pi} \int \frac{d^3 k}{\sqrt{(2\pi)^3 2 \hbar \omega_{\boldsymbol{k}}}} \left(a_{\boldsymbol{k}} e^{-i k \cdot x} + \boldsymbol{a}_{\boldsymbol{k}}^+ e^{i k \cdot x} \right)$$

where $a_{\boldsymbol{k}}$ is now a 4-vector &

$$\omega_{\boldsymbol{k}} = c \sqrt{\boldsymbol{k}^2 + \left(\frac{m_U c}{\hbar}\right)^2} \qquad k^\mu = \left(\frac{\omega_{\boldsymbol{k}}}{c},\ \boldsymbol{k} \right)$$

Since there're only 3 independent fields, we set

$$a_{\boldsymbol{k}\,\mu} = \sum_{\lambda=1}^{3} \varepsilon_\mu^\lambda(\boldsymbol{k})\, a_{\boldsymbol{k}}^\lambda$$

where ε_μ^λ are 3 orthonormal polarization vectors.

Since we're working in the Lorentz gauge,

$$k^\mu a_{\boldsymbol{k}\,\mu} = \sum_{\lambda=1}^{3} k^\mu \varepsilon_\mu^\lambda(\boldsymbol{k})\, a_{\boldsymbol{k}}^\lambda = 0 \qquad \forall\ \boldsymbol{k} \quad \& \quad k^\mu = \left(\frac{\omega_{\boldsymbol{k}}}{c},\ \boldsymbol{k}\right)$$

$$\rightarrow \qquad k^\mu \varepsilon_\mu^\lambda(\boldsymbol{k}) = 0 \qquad \forall\ \lambda,\ \boldsymbol{k}$$

Orthonormality means

$$\varepsilon_\mu^\lambda(\boldsymbol{k})\, \varepsilon^{\lambda'\,\mu}(\boldsymbol{k}) = \delta^{\lambda\lambda'}$$

Completeness is

$$\sum_{\lambda=1}^{3} \varepsilon^{\mu\,\lambda}(\boldsymbol{k})\, \varepsilon_\nu^\lambda(\boldsymbol{k}) = \delta_\mu^\nu - \frac{k_\mu k^\nu}{k^2} \qquad k^2 = \left(\frac{m_U c}{\hbar}\right)^2$$

or

$$\sum_{\lambda=1}^{3} \varepsilon_\mu^\lambda(\boldsymbol{k})\, \varepsilon_\nu^\lambda(\boldsymbol{k}) = \eta_{\mu\nu} - \left(\frac{\hbar}{m_U c}\right)^2 k_\mu k_\nu$$

As an example, for a massive boson moving along the z-axis,

$$k^\mu = \left(\frac{\omega_{\boldsymbol{k}}}{c},\ 0,\ 0,\ |\,\boldsymbol{k}\,|\right)^T$$

One correct choice of $\varepsilon_\mu^\lambda(\boldsymbol{k})$ is

$$\varepsilon^{\mu\,1}(\boldsymbol{k}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon^{\mu\,2}(\boldsymbol{k}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \varepsilon^{\mu\,3}(\boldsymbol{k}) = \frac{1}{k}\begin{pmatrix} |\,\boldsymbol{k}\,| \\ 0 \\ 0 \\ \omega_{\boldsymbol{k}}/c \end{pmatrix}$$

so that

$$\varepsilon_\mu^1(\boldsymbol{k}) = (0,\ -1,\ 0,\ 0) \qquad \varepsilon_\mu^3(\boldsymbol{k}) = (0,\ 0,\ -1,\ 0)$$

$$\varepsilon_\mu^3(\boldsymbol{k}) = \frac{1}{k}\left(\,|\,\boldsymbol{k}\,|,\ 0,\ 0,\ -\omega_{\boldsymbol{k}}/c\right)$$

For example, the completeness relation now reads

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{k^2}\begin{pmatrix} \boldsymbol{k}^2 & 0 & 0 & |\,\boldsymbol{k}\,|\,\omega_{\boldsymbol{k}}/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\,|\,\boldsymbol{k}\,|\,\omega_{\boldsymbol{k}}/c & 0 & 0 & -\omega_{\boldsymbol{k}}^2/c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} - \frac{1}{k^2}\begin{pmatrix} \omega_{\boldsymbol{k}}^2/c^2 & 0 & 0 & -\,|\,\boldsymbol{k}\,|\,\omega_{\boldsymbol{k}}/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ |\,\boldsymbol{k}\,|\,\omega_{\boldsymbol{k}}/c & 0 & 0 & \boldsymbol{k}^2 \end{pmatrix}$$

which is an identity.